A BEHAVIORAL MODEL OF ADAPTATION

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ABSTRACT. Adaptation refers to the process of changing behavior in response to a variation in the environment. We propose a model of an adaptive individual that contemplates two forces: on the one hand the individual benefits from adopting the ideal response to the new environment, but on the other hand, behavioral change is costly. We lay down the axiomatic foundations of the model. We then study two applications. The first studies a situation where ideal behavior depends on the response of another adaptive individual. The second analyzes the case where the ideal response is influenced by the strategic interaction in a cheap talk-like game.

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1. Introduction

Individuals often experience shocks in their environments. The standard approach in economics entails an immediate adaptation to the new situation. However, research in biology and psychology has long seen adaptation to changing environments as a gradual process of cumulated behavioral changes.[1] Building on this evidence, in this paper we offer, for the first time, a framework to study behavioral adaptation that explains why and how an individual may gradually adapt her behavior to new situations. In essence, we envision an individual that rationally understands the benefits of responding to the new situation, but for whom behavioral change is costly as it comes with the necessity of modifying well-formed and easy-to-apply internal routines. As a result, the individual must integrate both forces and gradually adapt.

Consider for instance an individual that has suffered a medical problem and must face a change of diet. Alternatively, think of changes in one's time allocation patterns after starting a new career or changing one's individual values upon entering a new society. In all these situations, there is an initial behavior and the individual is affected by an important shock in the environment that suggests a different ideal response. The question arises on how to move from the initial to the ideal response. On the one hand, the individual is reluctant to change because she suffers a cost whenever her responses deviate from past behavior. On the other hand, the individual would like to adapt immediately to the new ideal response, because not doing so comes with a

[1] Evolutionary biology focuses on the changes over the life time of species whereas developmental psychology deals with changes over the life time of an individual. Some classical references can be found in Helson (1964), Williams (1966), Valsiner and Connolly (2002) and Zeigler (2014).

cost. Therefore, the individual contemplates the two forces, and consequently selects a sequence of behaviors to rationally accommodate them.

We incorporate the previous considerations in the following simple model

$$\min_{\{x_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \delta^{t-1} \Big(\alpha \sum_{k=1}^{K} |x_{(t-1)k} - x_{tk}|^q + (1-\alpha) \sum_{k=1}^{K} |x_{*k} - x_{tk}|^q \Big),$$

where δ is the discount factor of the individual, α captures the relative weight of adaptation costs, and $(1-\alpha)$ represents the cost of deviating from the ideal response, x_*. Costs are modeled through polynomial distance functions, which is a standard way of introducing monotone and convex preferences. This is a simple model of adaptive behavior with several advantages.

First, the model is tractable, allowing us to completely determine which sequences of behavior can be the optimal response to some adaptation problem. The main result of the paper provides empirical content to the model. Theorem 1 axiomatically characterizes the model using two properties: A monotonicity property regulating the direction of change in behavior and a constant difference ratio property controlling the rate of change over different periods. In turn, this allows us to provide the unique solution to the adaptation problem, which facilitates the study of comparative statics and the analysis of welfare.

Second, the model is flexible enough to facilitate the study of adaptive behavior in a number of diverse settings. We illustrate this point by way of two applications of the model; strategic coordination and strategic communication. For the first application, we study the case where two adaptive individuals with different initial behaviors benefit from coordinating on a common behavioral code. Think, for example, of bilateral agreements where convergence in behaviors is desirable but affected by strategic considerations. We study the resulting dynamic game and, in Theorem 2, show the existence of a Markov-perfect equilibrium where individuals gradually adapt towards each other's behavior. In the second application, we introduce uncertainty on the ideal response in a setting where an informed sender passes a signal on the uncertain ideal response to an uninformed receiver. We understand the receiver as an adaptive individual, while the sender only cares about the deviations between the behavior of the receiver and the realized ideal response. For instance, suppose a doctor that, after a medical examination, finds that the patient should adopt a specific diet. The doctor understands well the adaptive behavior of the patient and may strategically communicate a non-ideal diet in order to manipulate this adaptive process. The patient, at the same time, may also anticipate the strategic considerations of the doctor, and behave accordingly. We show that the setting transforms into a cheap-talk game and, in Theorem 4, we characterize the partition equilibria of the game.

We close this introduction by relating our paper to the literature. Research on intertemporal choice with changing behavior can be found in habit formation/addiction models. Among these studies the closest to our approach is Becker and Murphy (1988).[2] They study a model of rational addiction that involves changes in behavior as the outcome of a utility maximization process with time consistent preferences. Becker and Murphy analyze the comparative dynamics of addiction in a perfect capital market

[2]Other important works include Pollak (1970), O'Donoghue and Rabin (1999), Gruber and Koszegi (2001), Bernheim and Rangel (2004) or Gul and Pesendorfer (2007).

environment assuming that instantaneous utility is a function of the current consumption as well as the stock of past consumption. The crucial difference with their paper is conceptual. In our setting we assume there is a given ideal response, which is either exogenously determined or is the result of some strategic interaction, allowing us to study the notion of adaptation to the ideal response, which is the key concept we study in the paper. In terms of results, we also diverge by providing characterizing properties of this type of behavior, as well as applications of our model to a variety of settings.

The cost of adaptation can also be related to the literature on changing tastes in dual-self models.[3] To see our model from this perspective, one may think of a short-sighted self that prefers to stick to the current behavior and a rational self that perfectly understands the benefits of immediate adaptation to the ideal response. Under this interpretation, and contrary to most of these models, the behavior of our agent is the result of an integration of these selves. In that sense, the closest paper to ours would be Loewenstein and O'Donoghue (2004) since their *'two minds'* model brings also a maximization problem of a weighted utility function. However their model lives in a static world and modeling dynamics is fundamental to understand the cost of adaptation.

Akerlof (1997) studies the case where an individual considers the average behavior of the individuals in their reference group, and either aims at minimizing the distance with average behavior, as in the case of conformism, or at positively deviating from it, as in the status seeking model. Hence, we share the view with Akerlof (1997)

[3]See the classical work of Strotz (1955) and also, Thaler and Shefrin (1981), Schelling (1982), Benabou and Pycia (2002), Fudenberg and Levine (2006), Pennesi (2021) and references therein.

in considering behavioral targets, that may have a social nature, as we discuss in Section 3.1. However, we deviate by focusing in the dynamic process of adaptation towards them. This dynamic process can also be interpreted as the *internalization* of prescriptions that is dictated by a new identity, as first formalized in economics by Akerlof and Kranton (2000). The target behavior then corresponds to the behavior associated with this identity and the optimal internalization process appears as the solution to the negotiation between initial preferences and the new set of values.

Finally, a direct application of our setting can be found in models of firm-level decision making with adjustment costs within the macroeconomics literature. Models with convex adjustment costs are utilized mainly to explain gradually adjusting firm-level decision variables observed in the data, such as factor demand in partial-adjustment models (Hamermesh and Pfann, 1996; Hall, 2004), or investment demand in general and in q-theoretic models (Gould, 1968; Treadway, 1969; Yoshikawa, 1980; Abel and Eberly, 1994). Certainly, our microeconomics perspective differentiates us from this literature. We provide an axiomatization of the model in terms of testable properties for the observable behavior, enabling the inference of unobservable parameters of the model, and offer applications in order to study strategic considerations related to the behavioral adaptation process.

The organization of the paper is as follows: Section 2 introduces the set-up as well as the characterization of the model. Section 3 is devoted to the two applications, strategic coordination and strategic communication, respectively. Section 4 concludes. All proofs are left to an appendix.

2. The Adaptation Problem

Denote by Δ the simplex of dimension K.[4] The initial behavior of the individual is denoted by $x_0 \in \Delta$, while the ideal response is denoted by $x_* \in \Delta$. We assume polynomial cost functions of degree $q > 1$. Given the relative weight of the adaptation cost with respect to the cost of deviating from the ideal response, $\alpha \in [0, 1]$, and the discount factor of the individual, $\delta \in (0, 1)$, our individual solves what we call the Adaptation Problem with parameters $(x_0, x_*, \alpha, \delta)$, i.e., the individual chooses a sequence of actions $\{x_t\}_{t=1}^{\infty}$ in the simplex solving

$$\min_{\{x_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \delta^{t-1} \Big(\alpha \sum_{k=1}^{K} |x_{(t-1)k} - x_{tk}|^q + (1-\alpha) \sum_{k=1}^{K} |x_{*k} - x_{tk}|^q \Big).$$

We now investigate the testable implications of the model, by describing the optimal sequences emerging from this cognitive process. Formally, we address the question of what are the properties of an observed sequence of behaviors, $\{x_t\}_{t=1}^{\infty}$, that indicate that the individual is optimally resolving an underlying Adaptation Problem. In other words, the question arises on whether there is an Adaptation Problem with parameters $(x_0, x_*, \alpha, \delta)$ such that the observed behavior $\{x_t\}_{t=1}^{\infty}$ by the analyst is the solution to the problem. It turns out that only two intuitive properties are necessary and sufficient to characterize such behavioral sequences. To present them, let D_{tk} denote the change in behavior at time t for component k, i.e. $D_{tk} = x_{tk} - x_{(t-1)k}$.

Monotonicity (Mon). For any k and for any $t > 1$: $D_{tk} > 0 \Rightarrow D_{(t+1)k} > 0$.

[4]To ease the exposition, we have chosen to work on a simplex, that intuitively captures the motivating examples in the introduction. However, the results of this section can be easily reproduced for other standard spaces, including $[0, 1]^K$, $\mathbb{R}_+^K$ or $\mathbb{R}^K$.

Mon simply states that if we observe individual behavior over one dimension adjusting in a certain direction, further adjustments in this dimension must have exactly the same structure.

Constant Difference Ratio (CDR). Let k, k' such that $\{x_{tk}\}$ and $\{x_{tk'}\}$ are not constant sequences. Then, for any $t, t' > 1$: $\frac{D_{(t+1)k}}{D_{tk}} = \frac{D_{(t'+1)k'}}{D_{t'k'}}$.

CDR requires, for all dimensions and periods, the same constant rate of behavioral change. In other words, there is no asymmetric adaptation pattern for different dimensions and the individual shows no accelerating or decelerating rates of adaptation.

Theorem 1. *The sequence* $\{x_t\}_{t=1}^{\infty} \in \Delta^{\infty}$ *satisfies Mon and CDR if and only if it is the (unique) solution of an Adaptation Problem.*

Theorem 1, and its proof, provides several important results: (i) it permits to delineate when the data available to the analyst, i.e. the sequence $\{x_t\}_{t=1}^{\infty}$, can be understood as the behavior of an individual solving an Adaptation Problem, (ii) it characterizes the solution to the Adaptation Problem, and (iii) when the individual behaves according to the model, it shows how to recover from the observed behavior the unobserved parameters $(x_0, x_*, \alpha, \delta)$ of the model. We now proceed to discuss in more detail each one of these points.

As for the first point, Theorem 1 establishes that a sequence of behaviors $\{x_t\}_{t=1}^{\infty}$ can be understood as the result of solving an Adaptation Problem if and only if the sequence satisfies properties Mon and CDR. Mon and CDR are simple properties, with the aim of facilitating the testability of the model, more than aiming to lay down normative properties on behavioral processes.

As for point (ii), Theorem 1 shows that the Adaptation Problem has a unique solution, that is of the form $x_t = \lambda(\alpha,\delta)x_* + (1-\lambda(\alpha,\delta))x_{t-1}$ for all t. The individual adapts at every period t a fraction $\lambda(\alpha,\delta) \in [0,1]$ of the distance between the current situation, x_{t-1}, and the ideal response x_*, a fraction that we call the *rate of adaptation*. This rate is dimension and time-invariant and depends uniquely on the parameters α and δ. From equation (5.3), it is straightforward to see that $\lambda(\alpha,\delta)$ is increasing in the discount factor δ. The intuition behind this idea is as follows: If the individual becomes more patient, future costs due to inadaptation become more relevant, leading the agent to reduce the amount of such future costs by being more adaptive at each period. Similarly differentiating equation (5.3) with respect to α, one can see that $\lambda(\alpha,\delta)$ is decreasing in α. The larger the cost of adaptation, the lower the individual adapts at each period of time. Obviously, closed-form solutions for $\lambda(\alpha,\delta)$ require specific values of q. For instance when we focus our attention on the common quadratic costs, i.e, $q=2$, we derive the following continuous function for $\lambda(\alpha,\delta)$:

$$\lambda(\alpha,\delta) = \begin{cases} 1 & \text{whenever } \alpha = 0, \\ \frac{(\alpha\delta-1)+\sqrt{1+2\alpha\delta-4\alpha^2\delta+\alpha^2\delta^2}}{2\alpha\delta} & \text{otherwise.} \end{cases}$$

As for the recoverability of the parameters, point (iii) above, Theorem 1 allows the analyst to pinpoint from the behavioral data $\{x_t\}_{t=1}^{\infty}$ the underlying parameters of the Adaptation Problem. Suppose that $\{x_t\}_{t=1}^{\infty}$ is a non-constant sequence. The initial response x_0 can be uniquely identified as $x_0 = \frac{x_1-\lambda x_*}{1-\lambda}$, with $\lambda = \frac{x_2-x_1}{x_*-x_1}$. The ideal response can also be uniquely identified as $x_* = \lim_t x_t$. In principle different values of α and δ may correspond to the same rate of adaptation λ. However identifying one of the parameters leads to a unique identification of the other. One could therefore use standard elicitation tests of the discount factor parameter to ultimately identify the adaptation cost parameter α.

We now briefly remark on welfare. In order to do so we derive the value function, given the solution to the Adaptation Problem. Note that the recursive substitution in the solution to the Adaptation Problem yields $x_t = (1-\lambda(\alpha,\delta))^t x_0 + (1-(1-\lambda(\alpha,\delta))^t)x_*$, for all $t > 0$, and hence, the one-period costs at t would be $\alpha\lambda(\alpha,\delta)^q(1-\lambda(\alpha,\delta))^{q(t-1)}\sum_{k=1}^K |x_{k0}-x_{*k}|^q$ and $(1-\alpha)(1-\lambda(\alpha,\delta))^{qt}\sum_{k=1}^K |x_{*k}-x_{k0}|^q$. The value function for $\lambda(\alpha,\delta) \in (0,1)$ would then become $V(x_0) = \sum_{t=1}^\infty \delta^{t-1}[\alpha\lambda(\alpha,\delta)^q(1-\lambda(\alpha,\delta))^{q(t-1)}\sum_{k=1}^K |x_{k0} - x_{*k}|^q + (1-\alpha)(1-\lambda(\alpha,\delta))^{qt}\sum_{k=1}^K |x_{*k} - x_{k0}|^q] = \frac{\alpha\lambda(\alpha,\delta)^q+(1-\alpha)(1-\lambda(\alpha,\delta))^q}{1-\delta(1-\lambda(\alpha,\delta))^q}\sum_{k=1}^K |x_{*k}-x_{0k}|^q = \omega(\alpha,\delta)\sum_{k=1}^K |x_{*k}-x_{0k}|^q$. This expression neatly reflects the behavioral loss of adapting behavior. It is multiplicatively separable, with two intuitive terms. The latter simply captures a measure of the gap between the initial and the ideal response. The former, a function of the behavioral parameters α and δ only, represents how costly it is for the individual to cover such gap. Let us focus again on the interesting case where the behavioral sequence is non-constant. It follows that higher valuation of the future results in larger welfare loss, i.e., $\frac{\partial\omega}{\partial\delta} > 0$. On the other hand, the effect of α on welfare loss is less obvious. Expressing δ as an implicit function of α and λ and substituting it in $\omega(\cdot)$ reveals that $\omega(\cdot)$ is equal to

$\alpha\lambda(\alpha,\delta)^{q-1}$ and, ultimately, allows to show that $\omega(\cdot)$ is strictly concave in α.[5] As a consequence, α has both a direct effect and an indirect effect through the adaptation rate. For lower levels of α, the direct effect outweights and welfare loss increases with α whereas for higher levels of α the opposite takes place.

3. Strategic Targeting

We have assumed so far that the individual internalizes an exogenous ideal response and attempts to adjust accordingly. In many economic situations, it happens to be the case that the ideal response emerges as the result of different strategic considerations. In this section, we study two applications where this is the case.

In the first subsection, we examine a situation where two adaptive individuals would like to coordinate on their behavior. However, since moving away from current behavior is costly, there are strategic considerations in the adaptation process of both individuals, resulting in a dynamic game where the optimal strategy for each individual depends on not only their own past behavior but also on the simultaneous behavior of the other individual. In the second subsection, we consider a situation where a second individual has private information about the uncertain ideal response of the decision maker and, having understood the adaptive nature of the decision maker, communicates this ideal response strategically. Our study of strategic communication, therefore, requires us to show how to incorporate the treatment of uncertainty in the precise level of ideal response. Unlike the first application this setting does not define a dynamic game,

[5]To see that, we first compute $\frac{\partial\lambda}{\partial\alpha}$ from equation (5.3) and substitute it in $\frac{\partial\omega}{\partial\alpha}$, before differentiating it once more.

instead it specifies a cheap-talk game where the response of the receiver has a dynamic nature. To simplify the exposition and the presentation of results, we assume in this section that the space of possible decisions is $[0,1]$ and that $q=2$.

3.1. **Strategic Coordination.** Consider a situation where two individuals with different initial behaviors would like to coordinate on a common behavioral code, yet changing their behavior is costly. It is in their interest to converge as much as possible on common rules, but they have well-established routines learnt from their personal experiences which hamper the adoption of a common standard. Certainly both of them would prefer the other individual to adopt their own standard immediately, but as long as there is a strictly positive cost to adaptation, this will never constitute an equilibrium behavior, as hinted by the baseline model. In this setting the ideal response is not exogenously given, but is dynamically determined by the interactive behavior with another individual. We investigate here the equilibrium dynamics of this situation.

Let (α^x, δ^x) and (α^y, δ^y) denote the personal characteristics of two adaptive individuals with initial behaviors given by x_0 and y_0, respectively. Let $\alpha^x, \alpha^y \in (0,1)$ and $x_0 \neq y_0$ to eliminate degenerate equilibria. A Coordination Problem given by the parameters $(x_0, \alpha^x, \delta^x; y_0, \alpha^y, \delta^y)$ refers to the case where each individual chooses a sequence of actions, $\{x_t\}_{t=1}^{\infty}$ and $\{y_t\}_{t=1}^{\infty}$ respectively, in order to minimize their corresponding life-time cost functions given by:

$$\sum_{t=1}^{\infty} (\delta^x)^{t-1} \Big(\alpha^x (x_{t-1} - x_t)^2 + (1-\alpha^x)(y_t - x_t)^2 \Big),$$

$$\sum_{t=1}^{\infty} (\delta^y)^{t-1} \Big(\alpha^y (y_{t-1} - y_t)^2 + (1-\alpha^y)(x_t - y_t)^2 \Big).$$

This setting now defines a dynamic game where the actions chosen by each individual determine the state for next period for both of them. Let $\sigma^i = (\sigma^i_t)_{t=1}^{\infty}$ denote a strategy for individual $i = x, y$.

In the following result we show that the strategy profile (σ^x, σ^y) defined by $\sigma^i_t(x_{t-1}, y_{t-1}) = \gamma^i j_{t-1} + (1-\gamma^i) i_{t-1}$ for $i, j \in \{x, y\}$ with $i \neq j$ for all $t > 0$ constitutes a Markov-perfect Nash equilibrium for a unique pair of $\gamma^x, \gamma^y \in (0,1)$. This equilibrium observationally happens to coincide with one where both individuals gradually adapt, *at the same rate of adaptation*, towards a common response.

Theorem 2. *The Coordination Problem* $(x_0, \alpha^x, \delta^x; y_0, \alpha^y, \delta^y)$ *has a Markov-perfect Nash equilibrium in which both individuals converge to* $\frac{\gamma^y}{\gamma^x+\gamma^y} x_0 + \frac{\gamma^x}{\gamma^x+\gamma^y} y_0$ *at a common rate of adaptation* $\gamma^x + \gamma^y$*, with* $\gamma^x, \gamma^y \in (0,1)$.

Theorem 2 shows that the strategic coordination problem of the two players aiming at adopting common standards has a simple, intuitive equilibrium. The common ideal response is the result of the linear combination of the initial responses, x_0 and y_0, using the coefficients $\frac{\gamma^y}{\gamma^x+\gamma^y}$ and $\frac{\gamma^x}{\gamma^x+\gamma^y}$, respectively. The scalars γ^x and γ^y can be obtained from equation (5.6) and the corresponding version for γ^y. In addition, Theorem 2 shows that both players adopt exactly the same rate of adaptation, given by $\gamma^x + \gamma^y$, towards this equilibrium behavior. From the point of view of testability, this equilibrium has the following properties on data: (i) Mon is satisfied by both individuals, (ii) CDR is satisfied by both individuals and also across individuals, and (iii) the two sequences converge to the same point.

Interestingly, note that the process of adaptation in this Coordination Game has characteristics of public goods settings. Moving away from the initial response entails a personal cost and at the same time generates a positive externality on the other player, as the distance between the behavior of the two players is reduced. This immediately implies that the equilibrium described will be inefficient; specifically, the maximization of aggregate welfare would require players to adapt more quickly than they are when they behave strategically. We illustrate this point for the symmetric case $\alpha_x = \alpha_y = \alpha$ and $\delta_x = \delta_y = \delta$. In this case it follows immediately that the common ideal response, both at equilibrium and the efficient one, is $\frac{x_0+y_0}{2}$. Therefore, we can focus on the comparison of the corresponding rates of adaptation. The optimality condition derived from the Bellman equation for the utilitarian problem would be $\alpha\gamma^x - 2(1-\alpha)(1-\gamma^x - \gamma^y) - \delta\alpha\gamma^x(1-\gamma^x-\gamma^y) = 0$. Since the symmetry implies $\gamma^x = \gamma^y = \gamma$, this equation can be written as

$$\alpha\gamma - 2(1-\alpha)(1-2\gamma) - \delta\alpha\gamma(1-2\gamma) = 0. \tag{3.1}$$

Then equation (5.5), the optimality condition from individual x's problem can be manipulated to resemble the former as

$$\alpha\gamma - 2(1-\alpha)(1-2\gamma) - \delta\alpha\gamma(1-2\gamma) + \{(1-\alpha)(1-2\gamma)[1+\delta\gamma(1-2\gamma)]\} = 0. \tag{3.2}$$

Now notice that the expression inside the paranthesis is strictly positive and the expression before that, which is basically the left hand side of equation (3.1), is a strictly increasing function of γ. Hence for all (α, δ), the γ that solves equation (3.1) is higher than the one that solves equation (3.2). The efficient rate of adaptation for both of the individuals, 2γ from (3.1), is always strictly greater than their equilibrium rate of adaptation, 2γ from (3.2).

3.2. **Strategic Communication.** Consider a situation in which an adaptive individual (from now on, the receiver) inquires of another individual (from now on, the sender), the ideal response before deciding on a sequence of behaviors. The sender can perfectly observe the ideal response, only cares about receiver's deviations from it and is fully aware of the adaptive nature of the receiver. Consequently, the sender has incentives to manipulate the observed ideal response, attempting to influence the adaptation of the receiver. Also, having anticipated the strategic intentions of the sender, the receiver may internalize a different ideal response to the communicated one. For example, consider the case in which a patient is the receiver, a doctor is the sender, and as a consequence of a health shock, a new ideal response is needed. The doctor hopes for an immediate adaptation of the patient to the new situation, but the gradually adaptive nature of the patient may delay the process in time. If the doctor believes that the patient is reluctant to adapt immediately, she may communicate a more extreme target than it is actually necessary, just to provoke behaviors that are closest to the ideal response. Similarly, the patient can anticipate the strategic intentions of the doctor and react accordingly by internalizing an ideal response different to the one that has been communicated.

We assume that the ideal response is governed by the continuous density function $f(x_*)$ with mean x_f and variance v_f. We model the receiver as an expected cost minimizer where costs are evaluated according to the function described in Section 2, and the sender's cost function as $\sum_{t=1}^{\infty} \delta_s^{t-1}(x_* - x_t)^2$, where δ_s is her discount factor. We also assume that $\alpha \in (0, 1)$, in order to focus on the case in which the sender and the receiver have only partially aligned preferences. All the aspects of the game are common knowledge except the realization of x_*.

After observing the ideal response, the sender passes a signal $\hat{x}$, which induces posterior beliefs about x^* for the receiver. The main difference with the baseline model from the perspective of the receiver is this uncertainty over the target behavior. Therefore, we first establish how the optimal behavior of the receiver changes under uncertainty before solving the communication game. The following theorem shows that at the presence of uncertainty about x^*, the receiver gradually adapts at a rate of adaptation $\lambda(\alpha, \delta)$ to the expected value of the ideal response. Suppose the receiver entertains posterior beliefs on the ideal response given by the density function $g(x_*)$, with mean x_g and variance v_g.

Theorem 3. *The solution to the Random Adaptation Problem $(x_0, g(x_*), \alpha, \delta)$, is $x_t = \lambda(\alpha, \delta)x_g + (1 - \lambda(\alpha, \delta))x_{t-1}$, for all $t > 0$, with $\lambda(\alpha, \delta)$ defined as in Section 2.*

Hence what matters for the receiver is the expected value of the ideal response. We assume that upon receiving the signal $\hat{x} \in [0, 1]$, the receiver uses Bayesian updating for her prior, i.e. $p(x_* \mid \hat{x}) = q(\hat{x} \mid x_*)f(x_*)/\int_0^1 q(\hat{x} \mid x)f(x)dx$. In order to be able to solve the equilibrium explicitly, let us assume that f is uniform. Then, following Crawford and Sobel (1982), a Bayesian Nash equilibrium of this game is a family of (possibly stochastic) signaling rules for the sender $q(\hat{x} \mid x_*)$, i.e. $\int_0^1 q(\hat{x} \mid x_*)d\hat{x} = 1$, and an action rule for the receiver $\{x_t(\hat{x})\}$ with the following properties: (i) for each

$x_* \in [0,1]$ any of the signals used with positive probability in $q(\hat{x} \mid x_*)$ minimizes the sender's cost given the action rule of the receiver and (ii) for each $\hat{x} \in [0,1]$, and taking the sender's signalling rule as given, the action $\{x_t(\hat{x})\}$ minimizes receiver's expected cost up on Bayesian updating.

The main difference from the classical cheap-talk setting is the gradual adaptation of the behavior. In the following theorem we deal with this by building on Theorem 3. We collapse the dynamics thanks to the recursive structure of the optimal action of the receiver and show that both cost functions satisfy the properties required for a partition equilibrium. To simplify the exposition, denote by $x_f(a,b)$ the mean of the ideal response, given the prior in the interval $[a,b]$, and by $\{x_t(a,b)\}$ the sequence converging to $x_f(a,b)$ given the rate of adaptation of the receiver, λ.

Theorem 4. *There exists an integer $\bar{N}$ such that, for every $1 \leq N \leq \bar{N}$, there exists $\{a_0, \ldots, a_N\}$ with $0 = a_0 \leq \cdots \leq a_N = 1$ for which a partition equilibrium $(\{x_t(\hat{x})\}, q(\hat{x} \mid x_*))$ exists, with: (i) $q(\hat{x} \mid x_*)$ is uniform, supported on $[a_i, a_{i+1}]$ if $x_* \in (a_i, a_{i+1})$, (ii) $\sum_{t=1}^{\infty} \delta_s^{t-1}(a_i - x_t(a_i, a_{i+1}))^2 = \sum_{t=1}^{\infty} \delta_s^{t-1}(a_i - x_t(a_{i-1}, a_i))^2$, (iii) $\{x_t(\hat{x})\} = \{x_t(a_i, a_{i+1})\}$ whenever $\hat{x} \in [a_i, a_{i+1}]$, and (iv) $a_{i+1} = Ha_i - a_{i-1} + 2 - H$, where $H(\alpha, \delta_r, \delta_s) = \frac{2(1-\delta_s(1-\lambda))(2-\lambda)}{\lambda(1+\delta_s(1-\lambda))}$ for $i < N$.*

Theorem 4 solves for the partition equilibria in the communication game, in which the receiver has adaptive behavior. Despite the misalignment of preferences, the sender is able to partially reveal information on the ideal response to the adaptive receiver. The equilibria work as follows: for a partition of $[0,1]$ into intervals, the sender merely communicates the interval to which the ideal response belongs, and the receiver gradually adapts, at rate of adaptation $\lambda(\alpha, \delta)$, to the expected value of the ideal response. In our doctor-patient example, the doctor would categorize signals into degrees of severity of the illness or diet requirements, and communicate to the patient the category to which her case belongs to.

4. Final Remarks

The purpose in this paper has been to offer a simple but functional theoretical framework to deal with the question of behavioral adaptation to changing environments, a feature that we believe is the norm rather than the exception in real settings. Inspired by research in psychology and biology, our model envisions adaptation as a gradual process of behavioral change.

We have axiomatically characterized the solution to the model. Namely, we have given simple behavioral conditions on sequences of choice that identify whether behavior can be understood as the result of behavioral adaptation as predicted by our model. Moreover, we have obtained the unique solution to the model and give the optimal rate of adaptation to the new ideal response. We have shown the comparative statics of the solution with respect to the parameters of the model, and have discussed behavioral welfare. All this analysis has allowed us to offer an in-depth understanding of the predictions and implications of our model.

It has been our aim, also, to show the versatility of the model. To do so we have developed two natural applications of the model involving strategic considerations. In the first application we study the case where two players would like to decide on a common ideal standard. We have shown that this case has characteristics of public goods settings, since moving away from one's initial behavior comes at a personal cost, but generates positive externalities on the other player. In the second application we model the strategic determination of an ideal response, in a cheap-talk like game setting. The receiver settles her ideal response as the result of the signal sent by a sender, taking into consideration the strategic incentives of the sender. With these two applications we have aimed at highlighting that our model of adaptive individuals allows to model strategic rational interaction, in familiar settings.

We believe that our framework will be useful in understanding a variety of new issues related to the very question of adaptation, and also in novel applications where adaptation is likely to play an important role. As for the former, we may envision adaptive individuals that revise their targets due to psychological considerations. For example, it seems plausible that achieving goals may have a reinforcement effect on the level of aspirations, whereas failing goals create a demeaning effect. It would be interesting to extend our model to such a setting. As for the latter, the consideration of adaptive individuals may be instrumental in settings like markets with advertisement, political competition with informational shocks, etc. Our model of behavioral adaptation may draw new light in these important settings, that may have important policy implications.

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5. Appendix

Proof of Theorem 1. ($\Leftarrow$) We start by solving the Adaptation Problem with parameters $(x_0, x_*, \alpha, \delta)$. Notice first that, trivially, whenever $\alpha = 0$, the unique solution is $x_t = x_*$ for all $t > 0$ and, whenever $\alpha = 1$, the unique solution is $x_t = x_0$ for all $t > 0$. We now consider the case in which $\alpha \in (0, 1)$. The Bellman equation of the Adaptation Problem can be written as:

$$V(x_{t-1}) = \min\{\alpha \sum_{k=1}^{K} |x_{(t-1)k} - x_{tk}|^q + (1-\alpha) \sum_{k=1}^{K} |x_{*k} - x_{tk}|^q + \delta V(x_t)\}$$

The first order conditions are:

$$\alpha q |x_{(t-1)k} - x_{tk}|^{q-1} \frac{|x_{(t-1)k} - x_{tk}|}{x_{(t-1)k} - x_{tk}} + (1-\alpha) q |x_{*k} - x_{tk}|^{q-1} \frac{|x_{*k} - x_{tk}|}{x_{*k} - x_{tk}} + \delta \frac{\partial V(x_t)}{\partial x_{tk}} = 0.$$

The Benveniste-Scheinkman conditions are:

$$\frac{\partial V(x_{t-1})}{\partial x_{(t-1)k}} = -\alpha q |x_{(t-1)k} - x_{tk}|^{q-1} \frac{|x_{(t-1)k} - x_{tk}|}{x_{(t-1)k} - x_{tk}}.$$

Iterating one more period and combining with the first order conditions result in:

$$\alpha |x_{(t-1)k} - x_{tk}|^{q-1} \frac{|x_{(t-1)k} - x_{tk}|}{x_{(t-1)k} - x_{tk}} +$$

$$(1-\alpha) |x_{*k} - x_{tk}|^{q-1} \frac{|x_{*k} - x_{tk}|}{x_{*k} - x_{tk}} = \delta \alpha |x_{tk} - x_{(t+1)k}|^{q-1} \frac{|x_{tk} - x_{(t+1)k}|}{x_{tk} - x_{(t+1)k}}. \tag{5.1}$$

Now, we guess the solution to be $x_t = \lambda x_* + (1-\lambda)x_{t-1}$ for some $\lambda \in (0,1)$. Let us verify this. Since the result is trivial whenever $x_{*k} = x_{0k}$, we have to consider the following cases:

$$x_{*k} > x_{0k} \Rightarrow |x_{(t-1)k} - x_{tk}| = -(x_{(t-1)k} - x_{tk})$$

$$|x_{*k} - x_{tk}| = x_{*k} - x_{tk} \qquad \text{for all } t.$$

$$x_{*k} < x_{0k} \Rightarrow |x_{(t-1)k} - x_{tk}| = x_{(t-1)k} - x_{tk}$$

$$|x_{*k} - x_{tk}| = -(x_{*k} - x_{tk}) \qquad \text{for all } t.$$

Hence, equation (5.1) can be written as:

$$-\alpha|x_{(t-1)k} - x_{tk}|^{q-1} + (1-\alpha)|x_{*k} - x_{tk}|^{q-1} = -\alpha\delta|x_{tk} - x_{(t+1)k}|^{q-1}. \tag{5.2}$$

Moreover, our guess also implies that $(x_{tk} - x_{(t-1)k}) = \lambda(x_{*k} - x_{(t-1)k})$ and $(x_{*k} - x_{tk}) = (1-\lambda)(x_{*k} - x_{(t-1)k})$ for all $t > 0$. Substituting these into (5.2), we arrive at:

$$-\alpha|-\lambda(x_{*k} - x_{(t-1)k})|^{q-1} +$$

$$(1-\alpha)|(1-\lambda)(x_{*k} - x_{(t-1)k})|^{q-1} = -\alpha\delta| - \lambda(1-\lambda)(x_{*k} - x_{(t-1)k})|^{q-1}$$

finally yielding

$$-\alpha\lambda^{q-1} + (1-\alpha)(1-\lambda)^{q-1} = -\alpha\delta\lambda^{q-1}(1-\lambda)^{q-1}$$

$$\lambda^{q-1}\Big[\frac{1}{(1-\lambda)^{q-1}} - \delta\Big] = \frac{1-\alpha}{\alpha}. \tag{5.3}$$

The left hand side of this equation is a continous and strictly increasing function of λ that approaches 0 and ∞ when λ approaches 0 and 1, respectively. Thus, the equality must hold for exactly one value $\lambda(\alpha,\delta)$, and we have verified the guess.

To summarize, notice that setting $\lambda(0,\delta)=1$ and $\lambda(1,\delta)=0$, we can then claim that for every $\alpha\in[0,1]$ and $\delta\in(0,1)$, $x_t=\lambda(\alpha,\delta)x_*+(1-\lambda(\alpha,\delta))x_{t-1}$ for all $t>0$ constitutes a solution to the Adaptation Problem. Uniqueness of this solution follows from the strict convexity of the one-period cost function and the fact that the domain of the decision variable is the unit interval.

We now show that the solution satisfies Mon and CDR. Notice that $\lambda(\alpha,\delta)\in\{0,1\}$ implies that $D_{tk}=0$ for all k and $t>1$ and both properties trivially hold. If $\lambda(\alpha,\delta)\in(0,1)$, it must be $D_{tk}\neq 0$ for any $t>1$ and we can obtain $\frac{D_{(t+1)k}}{D_{tk}}=\frac{x_{(t+1)k}-x_{tk}}{x_{tk}-x_{(t-1)k}}=\frac{\lambda(\alpha,\delta)\,x_{*k}+(1-\lambda(\alpha,\delta))x_{tk}-\lambda(\alpha,\delta)\,x_{*k}+(1-\lambda(\alpha,\delta))x_{(t-1)k}}{x_{tk}-x_{(t-1)k}}=\frac{(1-\lambda(\alpha,\delta))(x_{tk}-x_{(t-1)k})}{x_{tk}-x_{(t-1)k}}=1-\lambda(\alpha,\delta)>0$ for any k and for any $t>1$. This implies again both Mon and CDR and concludes the argument.

($\Rightarrow$) Let $\{x_t\}_{t=1}^{\infty}$ satisfy Mon and CDR. We first show that any sequence $\{x_{tk}\}$ is either constant or strictly increasing or strictly decreasing. Suppose first that, for a given k, there exists t such that $D_{tk}>0$. By Mon, it must also be $D_{(t+1)k}>0$. But then, by iterative application of CDR, $\frac{D_{3k}}{D_{2k}}=\cdots=\frac{D_{tk}}{D_{(t-1)k}}=\frac{D_{(t+1)k}}{D_{tk}}=\frac{D_{(t+2)k}}{D_{(t+1)k}}=\ldots$, we must have $D_{tk}>0$ for all $t>1$. Suppose now that, for a given k, there exists t such that $D_{tk}<0$. Then, given the fact that x_t belongs to the simplex, there exists another component k' such that $D_{tk'}>0$. By CDR, $\frac{D_{(t+1)k}}{D_{tk}}=\frac{D_{(t+1)k'}}{D_{tk'}}>0$, and we must have $D_{(t+1)k}<0$. By iterative application of CDR, $D_{tk}<0$ for all $t>1$. We have then proved that every sequence $\{x_{tk}\}$ is either constant or strictly increasing or strictly decreasing. Hence, for every k, since $\{x_{tk}\}$ is a monotone sequence in $[0,1]$, it must converge. We denote the limit of this sequence by x_{*k}.

We now show that $\{x_t\}_{t=1}^{\infty}$ is the solution of an Adaptation Problem. If $D_{tk}=0$ for all $t>1$ and for all k, we can simply define $x_0=x_1=x_*$ and $\{x_t\}_{t=1}^{\infty}$ will be the solution to $(x_0,x_*,1,\delta)$ for any $\delta\in(0,1)$. Then, suppose that there exists k such that $\{x_{tk}\}$ is not constant. Define $\lambda=\frac{x_{2k}-x_{1k}}{x_{*k}-x_{1k}}$. From the reasoning in the previous paragraph we know that $\{x_{tk}\}$ is strictly monotone, and hence $\lambda\in(0,1)$. We now show recursively that $x_{tk}=\lambda x_{*k}+(1-\lambda)x_{(t-1)k}$ holds for all $t>1$. By definition, $x_{2k}=\lambda x_{*k}+(1-\lambda)x_{1k}$. Assume that this holds for all values up to t. We show that it also holds for $x_{(t+1)k}$. First notice that $x_{3k}-x_{2k}=(1-\lambda)(x_{2k}-x_{1k})$. By CDR, $\frac{x_{(t+1)k}-x_{tk}}{x_{tk}-x_{(t-1)k}}=\frac{x_{3k}-x_{2k}}{x_{2k}-x_{1k}}=(1-\lambda)$ for all $t>1$. But then, $x_{(t+1)k}-x_{tk}=(1-\lambda)(x_{tk}-x_{(t-1)k})$. By the induction argument, we have $x_{(t+1)k}=(1-\lambda)x_{tk}-(1-\lambda)x_{(t-1)k}+x_{tk}=\lambda x_{*k}+(1-\lambda)x_{tk}$, as desired. For any other non-constant $\{x_{tk'}\}$, CDR also ensures that $x_{tk'}=\lambda x_{*k'}+(1-\lambda)x_{(t-1)k'}$ holds for every $t>1$. This is also trivially true for any other k' such that $\{x_{tk'}\}$ is constant. We then have found a value $\lambda\in(0,1)$ such that $x_{tk}=\lambda x_{*k}+(1-\lambda)x_{(t-1)k}$ holds for every k and every $t>1$. Defining $x_{0k}=\frac{x_{1k}-\lambda x_{*k}}{1-\lambda}$ guarantees that $x_1=\lambda x_*+(1-\lambda)x_0$. Hence, from the sufficiency part, we just need to find $\alpha\in(0,1)$ and $\delta\in(0,1)$ such that $\lambda=\lambda(\alpha,\delta)$ holds, as given by equation (5.3). Notice that $\lambda(\alpha,\delta)$ is an onto function. Then, fixing any $\delta\in(0,1)$, equation (5.3) defines $\alpha\in(0,1)$ uniquely, concluding the proof. ■

Proof of Theorem 2. Consider individual x. The period t pay-off depends on not only her previous action x_{t-1}, but also on the period strategy of the other player, which in turn is also a function of x_{t-1} as well as y_{t-1}, i.e., $y_t = \sigma_t^y(x_{t-1}, y_{t-1})$. We first show that the strategy profile (σ^x, σ^y) defined by $\sigma_t^i(x_{t-1}, y_{t-1}) = \gamma^i j_{t-1} + (1-\gamma^i)i_{t-1}$ for $i, j \in \{x, y\}$ with $i \neq j$ for all $t > 0$ is indeed a Markov-perfect Nash equilibrium for a unique pair of $\gamma^x, \gamma^y \in (0,1)$. The Bellman equation of the Coordination Problem for individual x can be written as:

$$V^x(x_{t-1}, y_{t-1}) = \min\{\alpha^x(x_{t-1} - x_t)^2 + (1-\alpha^x)(\sigma_t^y(x_{t-1}, y_{t-1}) - x_t)^2 + \delta^x V^x(x_t, y_t)\}.$$

The first order condition of the right hand side of this equation is:

$$-2\alpha^x(x_{t-1} - x_t) - 2(1-\alpha^x)(\sigma_t^y(x_{t-1}, y_{t-1}) - x_t) + \delta^x \frac{\partial V(x_t)}{\partial x_t} = 0,$$

and the first derivatives with respect to the state variables are given by:

$$\frac{\partial V^x(x_{t-1}, y_{t-1})}{\partial x_{t-1}} = 2\alpha^x(x_{t-1} - x_t) + 2(1-\alpha^x)(\sigma_t^y(x_{t-1}, y_{t-1}) - x_t)\frac{\partial \sigma_t^y(x_{t-1}, y_{t-1})}{\partial x_{t-1}}$$
$$\frac{\partial V^x(x_{t-1}, y_{t-1})}{\partial y_{t-1}} = 2(1-\alpha^x)(\sigma_t^y(x_{t-1}, y_{t-1}) - x_t)\frac{\partial \sigma_t^y(x_{t-1}, y_{t-1})}{\partial y_{t-1}}$$

Iterating the first expression one more period and combining it with the first order condition result in:

$$\begin{aligned}(5.4)\quad & -2\alpha^x(x_{t-1} - x_t) - 2(1-\alpha^x)(\sigma_t^y(x_{t-1}, y_{t-1}) - x_t) + \\ & \delta^x[2\alpha^x(x_t - x_{t+1}) + 2(1-\alpha^x)(\sigma_{t+1}^y(x_t, y_t) - x_{t+1})\frac{\partial \sigma_{t+1}^y(x_t, y_t)}{\partial x_t}] = 0,\end{aligned}$$

Now notice that for (σ^x, σ^y) as defined, we get $\sigma_t^y - \sigma_t^x = y_t - x_t = (1 - \gamma^x - \gamma^y)(y_{t-1} - x_{t-1}) = \cdots = (1-\gamma^x-\gamma^y)^t(y_0 - x_0)$. Second, notice also that $(x_{t-1} - x_t) =$

$\gamma^x(x_{t-1} - y_{t-1}) = -\gamma^x(1-\gamma^x-\gamma^y)^{t-1}(y_0 - x_0)$. Since, $\frac{\partial \sigma^y_{t+1}(x_t,y_t)}{\partial x_t} = \gamma^y$, equation (5.4) becomes:

$$2\alpha^x\gamma^x(1-\gamma^x-\gamma^y)^{t-1}(y_0-x_0) - 2(1-\alpha^x)(1-\gamma^x-\gamma^y)^t(y_0-x_0)+$$
$$\delta^x[-2\alpha^x\gamma^x(1-\gamma^x-\gamma^y)^t(y_0-x_0) + 2(1-\alpha^x)(1-\gamma^x-\gamma^y)^{t+1}(y_0-x_0)\gamma^y] = 0,$$

or simply,

$$\alpha^x\gamma^x - (1-\alpha^x)(1-\gamma^x-\gamma^y) + \delta^x[-\alpha^x\gamma^x(1-\gamma^x-\gamma^y) + (1-\alpha^x)\gamma^y(1-\gamma^x-\gamma^y)^2] = 0. \tag{5.5}$$

To obtain equation (5.5), we have used the fact that $\gamma^x + \gamma^y \neq 1$. This follows from the fact that the individual faces non-zero routine costs, $\alpha^x > 0$, which leads her not to adapt completely after one period. Now, equation (5.5) can be rewritten as a second-degree equation on γ^x as a function of $\alpha^x, \delta^x, \gamma^y$:

$$\delta^x[\alpha^x(1-\gamma^y)+\gamma^y](\gamma^x)^2 + [1 - 2(1-\alpha^x)\delta^x\gamma^y(1-\gamma^y) - \delta^x(1-\gamma^y)\alpha^x]\gamma^x + (1-\alpha^x)(1-\gamma^y)[\delta^x\gamma^y(1-\gamma^y)-1] = 0. \tag{5.6}$$

The negative root of this expression is always smaller than 0, hence $\gamma^x(\alpha^x, \delta^x, \gamma^y)$ would be the positive root, that always lies in $(0,1)$ (the explicit expression is omitted as it is cumbersome). Notice that the solution to individual y's problem would yield $\gamma^y(\alpha^y, \delta^y, \gamma^x) \in (0,1)$ in a similar way. The uniqueness of the pair (γ^x, γ^y) follows from the fact that both of these functions are continuous in $(0,1)$ and strictly decreasing in γ^y and γ^x, respectively. Also, notice that, in consonance with the adaptation rate obtained in Theorem 1, the limits of γ^x when γ^y approaches to 0 and 1 are, respectively, $\lambda(\alpha^x, \delta^x) < 1$ and 0. Similarly, the limits of γ^y when γ^x approaches 0 and 1 are, respectively, $\lambda(\alpha^y, \delta^y) < 1$ and zero. Thus, for given $(\alpha^x, \alpha^y, \delta^x, \delta^y)$, the reaction functions must cross (only once) and hence, there exists a unique pair $(\gamma^x, \gamma^y) \in (0,1)$.

For the pair of strategies in equilibrium, (γ^x, γ^y), notice that individual x converges to $x_0 + \sum_{t=1}^{\infty}(x_t - x_{t-1}) = x_0 + \sum_{t=1}^{\infty} \gamma^x(1-\gamma^x-\gamma^y)^{t-1}(y_0 - x_0) = x_0 + \frac{\gamma^x(y_0-x_0)}{\gamma^x+\gamma^y} = \frac{\gamma^x y_0 + \gamma^y x_0}{\gamma^x+\gamma^y} = \frac{\gamma^x}{\gamma^x+\gamma^y}y_0 + \frac{\gamma^y}{\gamma^x+\gamma^y}x_0$, as desired. With respect to the rate of adaptation, notice that $x_{t+1} - x_t = \gamma^x(1-\gamma^x-\gamma^y)^t(y_0-x_0) = (1-\gamma^x-\gamma^y)(x_t - x_{t-1})$ and hence, from the reasoning in Theorem 1, the rate of adaptation of x is equal to $\gamma^x + \gamma^y$. The same holds for individual y, concluding the proof. ■

Proof of Theorem 3. The receiver minimizes the expected cost function $\mathbb{E}[\sum_{t=1}^{\infty}\delta^{t-1}(\alpha(x_{t-1} - x_t)^2 + (1-\alpha)(x_* - x_t)^2)]$, which can be equivalently expressed as follows:

$$\begin{aligned}
\mathbb{E}[\cdot] &= \sum_{t=1}^{\infty}\delta^{t-1}\alpha(x_{t-1}-x_t)^2 + (1-\alpha)\mathbb{E}[\sum_{t=1}^{\infty}\delta^{t-1}(x_*-x_t)^2] \\
&= \sum_{t=1}^{\infty}\delta^{t-1}\alpha(x_{t-1}-x_t)^2 + (1-\alpha)\mathbb{E}[\sum_{t=1}^{\infty}\delta^{t-1}(x_*-x_g+x_g-x_t)^2] \\
&= \sum_{t=1}^{\infty}\delta^{t-1}\alpha(x_{t-1}-x_t)^2 + (1-\alpha)\mathbb{E}[\sum_{t=1}^{\infty}\delta^{t-1}((x_*-x_g)^2+(x_g-x_t)^2+2(x_*-x_g)(x_g-x_t))] \\
&= \sum_{t=1}^{\infty}\delta^{t-1}\alpha(x_{t-1}-x_t)^2 + (1-\alpha)[\sum_{t=1}^{\infty}\delta^{t-1}(v_g+(x_g-x_t)^2+\mathbb{E}[2(x_*-x_g)(x_g-x_t)])].
\end{aligned}$$

Now, notice that $\mathbb{E}[2(x_*-x_g)(x_g-x_t)]$ must be equal to zero, and hence, we can conclude that $\mathbb{E}[\sum_{t=1}^{\infty}\delta^{t-1}(\alpha(x_{t-1}-x_t)^2+(1-\alpha)(x_*-x_t)^2)] = \sum_{t=1}^{\infty}\delta^{t-1}(\alpha(x_{t-1}-x_t)^2+(1-\alpha)(x_g-x_t)^2)+(1-\alpha)\frac{v_g}{1-\delta}$. It is obvious from this expression that the expected cost function is equivalent to the cost function of the baseline model with the expected value of the ideal response x_g, plus an additional component that does not depend on x_t. Thus the solution follows immediately from Theorem 1 as $x_t = \lambda(\alpha,\delta)x_g + (1-\lambda(\alpha,\delta))x_{t-1}$, for all $t > 0$. ■

Proof of Theorem 4. From Theorem 3, we know that given $g(x^*)$, the cost minimizing action for the receiver is to choose $\{x_t\}$ such that $x_t = \lambda x_g + (1-\lambda)x_{t-1}$, for all $t > 0$. By using this recursive process and the fact that $\alpha, \delta, \lambda \in (0,1)$, we can rewrite the cost functions of both the sender and the receiver as a sole function of x_g instead of using the entire sequence of behaviors. Formally:

$$\begin{aligned}
C_s(x_g, x_*; x_0, \alpha, \delta, \delta_s) =& \frac{(x_* - x_g)^2}{1-\delta_s} + \frac{2(1-\lambda)(x_* - x_g)(x_g - x_0)}{1-\delta_s(1-\lambda)} + \frac{(1-\lambda)^2(x_g - x_0)^2}{1-\delta_s(1-\lambda)^2}, \\
C_r(x_g, x_*; x_0, \alpha, \delta) =& \frac{(1-\alpha)(x_* - x_g)^2}{1-\delta} + \frac{2(1-\alpha)(1-\lambda)(x_* - x_g)(x_g - x_0)}{1-\delta(1-\lambda)} \\
&+ \frac{[\alpha\lambda^2 + (1-\alpha)(1-\lambda)^2](x_g - x_0)^2}{1-\delta(1-\lambda)^2}.
\end{aligned}$$

These functions are twice continuously differentiable functions with $\frac{\partial^2 C_i}{\partial x_g^2} > 0$ and $\frac{\partial^2 C_i}{\partial \bar{x} x_g} < 0$, $i \in \{s, r\}$. Therefore, the existence of the equilibrium as described follows from Theorem 1 of Crawford and Sobel (1982). Having received the message that the ideal response belongs to (a_i, a_{i+1}), the posterior on this interval will also be uniform and the receiver will gradually adapt to the point $\frac{a_i + a_{i+1}}{2}$. The intervals can be determined through the arbitrage condition $\sum_{t=1}^{\infty} \delta_s^{t-1}(a_i - x_t(a_i, a_{i+1}))^2 = \sum_{t=1}^{\infty} \delta_s^{t-1}(a_i - x_t(a_{i-1}, a_i))^2$. That is, if the signal belongs to the boundary of two intervals, the sender must be indifferent between the gradual adaptation to the low or the high interval. Using the uniformity assumption, algebraic manipulation shows that this simply becomes a second degree difference equation $a_{i+1} = Ha_i - a_{i-1} + 2 - H$, where $H = \frac{2(1-\delta_s(1-\lambda))(2-\lambda)}{\lambda(1+\delta_s(1-\lambda))} > 2$. Hence, the solutions to the difference equation are of the form $k_1 r_1^i + k_2 r_2^i + x_0$, where $r_1 = \frac{H+\sqrt{H^2-4}}{2}$ and $r_2 = \frac{H-\sqrt{H^2-4}}{2}$ are the different real roots of the associated characteristic function. Since $a_0 = 0$ and $a_N = 1$, the constants become $k_1 = \frac{(1-r_2^N)x_0-1}{r_2^N-r_1^N}$ and $k_2 = \frac{1-(1-r_1^N)x_0}{r_2^N-r_1^N}$, providing ultimately the complete partition with N intervals. ■