BEHAVIORAL INFLUENCE*

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Abstract. In the context of stochastic choice, we introduce an individual decision model which admits a cardinal notion of peer influence. The model presumes that individual choice is not only determined by idiosyncratic evaluations of alternatives but also by the influence from the observed behavior of others. We establish that the equilibrium defined by the model is unique, stable and falsifiable. Moreover the underlying preference and influence parameters as well as the structure of the underlying network are uniquely identified from, arguably, limited data. The baseline model includes two individuals with conformity motives. Generalizations to multi-individual settings and negative interactions are also introduced and analyzed.

Keywords: Identification of social interactions, social influence, peer effects, stochastic choice, conformity, negative influence.
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1. Introduction

It is a well-established fact that individual choices are directly influenced by the choices of one's peers.[1] Identification of peer influence out of observable behavior has

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[1] There is an abundance of evidence corroborating peer influence in a variety of social contexts: Peer behavior has a significant influence not only on a student's school achievement (Calvo-Armengol et al., 2009), but also on social behavior such as consumption of recreational activities, drinking, smoking,

been a challenging problem for social scientists for decades.[2] At the heart of this issue lies Manski's reflection problem (Manski, 1993): Behavioral similarities among peers can be caused by correlated unobserved or observed characteristics as well as peer influence. Distinguishing between these effects is not straightforward due to the simultaneity in the behavior of interacting individuals. This paper provides a novel approach to the identification of peer influence by focusing on the microfoundations of interaction, rather than adopting ex-post estimation techniques. We introduce a simple model of decision making for interacting individuals that enables inference of underlying unobserved parameters out of observable behavior.

The novelty in our approach lies in the introduction of a new source of variation for social interaction models. Specifically, we vary the set of available options from which individuals choose. Without any variation in the choice set, the reflection problem cannot be solved. However with minimal variation, *e.g.*, observations from two choice sets rather than one, it becomes possible to identify the social influence. For instance consider two friends, Dan and Bob, and their choices on daily exercise routines during a countrywide lockdown due to a pandemic. Under strict government rules they can choose to either exercise home or go for a walk outside. Let Dan choose to exercise home 71% of the time and go for a walk for 29%, whereas these frequencies be 78% and 22% for Bob, respectively. Reflection problem emerges exactly at this point, where an outside observer cannot tell whether these friends are behaving similarly because they are influencing each other or they indeed have similar preferences (and/or backgrounds) and hence they would have also behaved the same way without interaction. Without any further information, an outside observer cannot differentiate between these two scenarios. In order to overcome the reflection problem and identify peer effects out

etc. (Sacerdote, 2011). High productivity co-workers are found to increase one's own productivity (Mas and Moretti, 2009). Involvement in crime (Glaeser et al., 1996), job search (Topa, 2001), adolescent pregnancy (Case and Katz, 1991), college major choice (De Giorgi et al., 2010) are other prominent examples in which social interactions are shown to be crucial constituents of individual behavior.

[2]See Blume et al. (2011); Bramoullé et al. (2020) for an early and recent reviews of literature, respectively.

of observable behavior, our methodology suggests to exploit the changes in these individuals' behaviors over a new choice set. For instance; when the lockdown is over and the updated government regulations allow also for exercising in the gym. These individuals' behaviors under these two scenarios, i.e., {exercise home, go for a walk} and {exercise home, go for a walk, go to gym}, are sufficient for our identification strategy to identify the peer effects as long as the observed choices are consistent with our model, as we will illustrate after introducing the model briefly.

Our main contribution is to provide an intuitive and tractable decision model which affords a meaningful, and measurable, definition of "influence" as derived from choice behavior alone. Our model consists of two essential parameters: An individual preference parameter and an individual influence parameter. The latter captures interdependence of behavior across individuals and can accommodate different values for different peers, enabling heterogeneity of peer effects. The individual preference parameter is more standard. It can be interpreted as the intrinsic utility of the underlying alternatives; the subjective value of the alternative absent any social effects.[3] Social influence transpires through the *observed behavior* of the other individual(s), where subjective value of each alternative is adjusted by a weighted version of the observed behavior of others regarding that alternative. As a result of this weighted aggregation process, individual choice behavior reflects the relative utility of each alternative in a given menu altered by social influence. More precisely, the choice frequency of each alternative from a menu is equal to the relative utility of this alternative under social influence, with respect to all other available alternatives.

Our identification strategy exploits the change in choice frequencies when a new alternative is introduced in order to pin down the peer influence and the underlying preferences. Going back to the example on exercise behavior, let us observe that once the government regulations allow for going to the gym, Dan and Bob's behaviors change as follows: Dan exercises home 60% of the time, goes for a walk for 26% and goes to the gym for 14%, whereas these frequencies are $70\%, 19\%, 11\%$ for Bob, respectively. This

[3] In social interactions literature, the non-influence parameters that affect individual behavior are defined via types of variables such as predetermined social factors including gender, age, race, etc. Our model abstracts away from these effects, classifying them under the individual preference parameter.

pair of behaviors are consistent with our model (as described in subsection 2.3), hence we can reveal the underlying preferences and the interaction parameters uniquely, overcoming the reflection problem. Interestingly, our identification strategy (as described in subsection 2.2) implies that although Dan and Bob's choice frequencies are aligned over exercise options, their idiosyncratic preferences are not aligned. For Bob, indeed the intrinsic utility of home exercise is the highest and going for a walk is the lowest, whereas for Dan, the exact opposite holds. However Bob's behavior has great influence on Dan. To be precise, conformity with Bob's behavior is five times more important to Dan than his own subjective evaluation, whereas for Bob his own evaluation and Dan's behavior are equivalently important.[4] Thus thanks to our identification strategy, we can deduce that strong conformity motives have resulted in the observed behavior.

Our model is a stochastic choice model that assumes consistent behavior across all budget sets. Critically, this menu variability grants us unique identification (or point identification, as coined in the econometrics literature). Moreover, our identification strategy does not suffer from a common handicap of identification in revealed preference or decision-theoretic models: arguably unrealistic data requirements. Many choice theoretic models require a rich dataset, typically individual choices from all menus, for identification purposes.[5] As we show in Subsection 2.2, observations from only two menus are sufficient for unique identification for our baseline model, involving two individuals. For identification of influence networks involving more than two individuals, observations from two menus can still be sufficient as long as there are sufficiently many alternatives in the menus. We elaborate more on this in Section 3.

We establish in subsection 2.3 that our model is falsifiable, by providing its empirical content in terms of choice. Three behavioral properties are sufficient to characterize the model. All of these properties are built around a cross-elasticity type parameter that evaluates the relative rate of change in the individual choice frequency of an alternative

[4]Subsection 2.1 introduces our model formally, but as described above two critical parameters constitute the primitives. In this example, the preference parameters, the intrinsic utility weights of exercise home, go for a walk, go to gym, for Dan and Bob are $0.1, 0.3, 0.6$ and $0.8, 0.08, 0.12$, respectively, with corresponding interaction parameters 5 and 1.

[5]For a recent exception to this common trend as well as a discussion on the topic, see Dardanoni et al. (2020).

as a response to a comparative change in the behavior of the other individual(s). In contrast to standard models of individual choice, this influence parameter is derived from the choice behaviors of *all* of the individuals jointly as opposed to the behavior of only one individual. Hence, these characterizing properties are entirely novel.

The parameters of our model define an "equilibrium," where the choice behavior of each individual is a function of idiosyncratic utility and influence parameters; as well as the behavior of the other individual(s). Unlike many other discrete choice models (Brock and Durlauf, 2001; Blume et al., 2011), the equilibrium defined by our model is unique. Moreover, it is also stable, in the sense that a dynamic adjustment procedure always tends to this unique equilibrium. In other words, if we believe that each individual aggregates behaviorally according to our procedure, we should expect their behavior to conform to our model in the long-run. There are two critical implications of this result. The first implication is more practical: if one individual mistakenly chooses, or one of them misobserves the other's choices at some period in time, their behavior will still revert to the predictions of our model in the long run. Second and more importantly, identification of the underlying parameters from dynamic data is also possible. Then in the absence of equilibrium choice behavior, we can use a similar identification strategy over consecutive choice data. Subsection 2.4 elaborates on this.

Our baseline model involves two individuals with conformity motives, as in the example above. An action's choice probability increases as the action is chosen more frequently by one's peer. However our model easily adapts to more individuals and accommodates other types of interaction. We present two simple extensions. The first incorporates multi-individual interaction, where individuals have different degrees of influence on the behaviors of their peers, and second "negative" influence, where the choice probability decreases as it is chosen more frequently by some peers.

We provide three distinct and well-known social influence settings where the behavior produced by our model can be reproduced under certain assumptions. We refer to these as 'foundational justifications' for our model since each of them can be seen as an economic mechanism underlying our model of influence. The first of those incorporates strategic interactions, introducing a simultaneous game setting whose Quantal

Response Equilibrium happens to coincide with our model, whereas the second one is utility maximization in a discrete choice setting with peer effects. The last mechanism is a basic naive learning set up as in DeGroot (1974). All of these models are distinguished from our model as we use menu variability in our setting.

The organization of the paper is as follows. The next subsection is devoted to literature review. Section 2 presents a detailed analysis of the baseline model with two individuals with conformity motives, including identification, falsifiability and stability results, as well as the foundational justifications. Section 3 introduces the generalization to multi-individual settings, whereas Section 4 incorporates negative influence to these settings. Finally, we conclude. All proofs are left to an appendix.

1.1. **Related Literature.** Economics research on identification of social interactions has mainly utilized econometrics tools and techniques. Most of these studies employ *linear social interaction models* (Manski, 1993; Blume et al., 2011; Jackson, 2011; Blume et al., 2015), where individual utility of an action is defined as a linear additive function with two components: an individual private utility and a social utility. Blume et al. (2015) provide micro-foundations to these linear interaction models by showing that under certain parametric assumptions they can be reproduced as the Bayesian-Nash equilibrium of an incomplete information game where individuals choose an action to maximize their expected utility given their type and the public types of others.[6] Calvo-Armengol et al. (2009) investigate the effect of the structure of social network and show that an underlying peer effects game rationalizes individual outcomes, where at the Nash equilibrium each outcome is proportional to the centrality of the individual within the network.

Linear social interaction models are defined for continuous choice variables. An alternative to this is developed by incorporating the linear additive utility function with interaction effects into a discrete choice setting (Blume, 1993; Brock and Durlauf, 2001, 2006). Binary or multinomial discrete choice models with social interactions make use of random fields models to study the equilibrium. Three critical assumptions ensure tractability of the model. First, the assumption of constant strategic complementarity:

[6]For identification strategies without parametric assumptions, see Brock and Durlauf (2007).

the cross-partial of social utility is a positive constant that is the same for all individuals. Second, rational expectations: the expected average behavior is simply the objective average behavior. Finally, the error terms follow a relevant extreme value distribution. These assumptions are sufficient to produce individual choice outcomes that are consistent with logistic choice with multiple equilibria. The majority of these papers assume large populations in order to justify the assumption that each individual ignores the effect of their own choice on the average choice of the society. An exception to this is Soetevent and Kooreman (2007), where they consider interaction in small groups in which choices of other individuals is fully observable. Thus, the choice of an individual directly depends on the observed behavior of the others. Our model also uses this intuition. Indeed, under certain assumptions the behavior produced by a multinomial discrete choice model with social interactions coincides with the behavior produced by our baseline model. This requires a different error distribution then the one commonly assumed for those works. We clarify this connection in subsection 2.5.

In this strand of literature social interactions has typically been taken to be generated by group specific averages. Incorporating network theory in the study of identification of social interactions has enabled a much richer analysis of the microstructure of interactions. Early works on this assumed a *known* network structure, based on common observables or self-reported, elicited data (Bramoullé et al., 2009; Lee et al., 2010; De Giorgi et al., 2010). However both of these methods bear shortcomings for econometric methods or practical reasons related to collecting data (De Paula, 2017). A first improvement on this was suggested by Blume et al. (2015) by assuming only partial information on the structure of the underlying network. De Paula et al. (2019) advances on this by assuming no a priori information on the network structure and provides sufficient conditions for full identification of social interactions with panel data. Our paper is complementary to this literature since our general model also encompasses an influence network, where the structure of the relations do not need to be known a priori. Instead it is fully revealed by the behaviors thanks to our identification strategy.

It is important to note that many theoretical models for identification of peer influence are restricted by strategic complementarity (Blume, 1993; Brock and Durlauf, 2001, 2006; Blume et al., 2011):[7] individual utility over an action increases with the number of peers taking the action, explaining mostly conformity-type behavior. However empirical evidence points out to negative interactions as well. For instance, Glaeser et al. (1996) suggests the existence of negative interactions among criminals due to competition for resources. Bhatia and Wang (2011) study peer effects in physician's prescription behavior and find a significantly negative effect on each other's prescription behavior, partly explained by observational learning and congestion effects. Foster and Rosenzweig (1995) find evidence of negative relation between experimental technology adoption rates of farmers and their neighbors. As we show in an extension, our model is flexible enough to accommodate negative interactions.

The use of choice theoretic tools to study social interactions is quite recent. As far as we know the first choice-theoretic work investigating influence across individuals is Cuhadaroglu (2017). This work introduces a deterministic model of two stage optimization where the first stage involves maximization of own preferences (transitive but not necessarily complete), and the second stage accommodates social influence to further refine first stage outcomes. Recently, two contemporaneous studies incorporate choice theoretic analysis to identification of peer effects. Borah and Kops (2019) and Kashaev and Lazzati (2019) both propose decision procedures in group settings that makes use of 'a consideration set' approach. Borah and Kops (2019) proposes a two stage mechanism, where the first stage is devoted to the formation of consideration sets with those alternatives that are chosen sufficiently enough by the members of peers and the second stage is devoted to preference maximization. Kashaev and Lazzati (2019) incorporate random consideration sets to the dynamic model of social interactions of Blume (1993). The main difference of our work from these models is about the channel through which others' behavior influence the individual. Our model presumes that social influence alters one's behavior via preferences, whereas those two papers assume a limitation of the choice set due to social influence.[8]

[7]Exemptions to this include structural models to identification such as Bramoullé et al. (2009); Cohen-Cole et al. (2018). Bramoullé (2007) studies the effect of the structure of the network on equilibrium behavior for games of anti-coordination, where there are incentives to anti-coordinate.

[8]Many findings from social psychology or experimental economics literatures support the notion that social influence alters one's preferences. For instance, Kremer and Levy (2008) show that alcohol

Fershtman and Segal (2018) also consider a social interaction set up where individual behavior not only depends on one's own preferences but also on the behavior of other agents in an expected utility framework. A social influence function converts the private utility of the agent and the observable utilities of everyone else to an observable utility for the agent. They study certain properties of social influence functions and their implications for the equilibrium without proposing an explicit behavioral model.

Finally, our work is related to the literature discussing the revealed preference implications of solution concepts in games; for example, Sprumont (2000); Lee (2012). One interpretation of the mathematics of our model is formalizing, for each choice set, a game and a solution concept. Thus, our model provides observable predictions of our concept as strategy sets vary. The aforementioned papers also study the predictions of game theory as strategy sets vary. In a similar fashion, our work is also linked to the literature on estimation and inference in discrete games; with the main difference being that rather than relying on parametric or structural estimation techniques, our main tool of inference is revealed preference. For early works on estimation in discrete games see Bresnahan and Reiss (1991); Kooreman (1994); for inference in large discrete games see Menzel (2016); for non-parametric estimation in non-cooperative games see Haile and Tamer (2003).

2. Behavioral Influence

2.1. **The Model.** Let X be a finite set of *alternatives* with $|X| > 2$. A *stochastic choice rule* is a map $p : 2^X \setminus \{\emptyset\} \to \cup_{S \subseteq X} \Delta_{++}(S)$ such that for all $S \subseteq X$, $p(S) \in \Delta_{++}(S)$.[9]

We propose a simple model of influence. There are two individuals, 1 and 2. Each individual is influenced by the choices of the other individual. The observable behavior

consumption by one roommate is more likely to influence the alcohol consumption of another roommate via a preference change rather than a modification of the choice set. According to the notion of (mis)identification in social psychology, when some alternatives become identified with certain identities, they become more likely to be preferred by aspiring individuals, whereas despising individuals avoid them in order not to be misindentified (Berger, 2016).

[9]The notation Δ_{++} refers to the set of probability distributions with full support. We denote $\sum_{x \in S} f(x)$ by $f(S)$ for any function f on X.

is a pair of stochastic choice rules (p_1, p_2) where p_i stands for individual i's choices. We use the notation $i, j \in \{1, 2\}$ with $i \neq j$ for the individuals in general. Then $p_i(x, S)$ stands for the probability of individual i choosing alternative x from S, certainly with $\sum_{x \in S} p_i(x, S) = 1$.

The primitives of our setting are idiosyncratic weights and influence parameters. Let $w_i \in \Delta_{++}(X)$, so that $w_i(x)$ measures the idiosyncratic weight of the available alternatives for individual i. These can be interpreted as intrinsic utilities of the alternatives absent any social influence effects as in the Luce model.[10] We postulate that the choice behavior of individual j regarding an alternative $x \in S$ directly influences individual i's evaluation of that alternative for the same choice set. Specifically we assume the utility of agent i from choosing alternative x from budget S is given by:

$$w_i(x) + \alpha_i p_j(x, S) \tag{1}$$

where α_i measures the degree of influence of j on i. For the baseline model, we assume that $\alpha_i \geq 0$, hence α_i acts as a conformity parameter. The higher the probability that j chooses x from S, the higher is i's evaluation of x in S. The value of x is influenced by the choice probability of others in a linear fashion. Hence, our formulation is in line with the classical linear interaction models such as Manski (1993); Blume et al. (2011, 2015). The choice probabilities are given by the normalized utility values as in the Luce model. Formally,

Definition. (p_1, p_2) has a **dual interaction** representation if there exist $w_1, w_2 \in \Delta_{++}$ and $\alpha_1, \alpha_2 \in \Re^+$ such that

$$p_i(x, S) = \frac{w_i(x) + \alpha_i p_j(x, S)}{\sum_{y \in S} [w_i(y) + \alpha_i p_j(y, S)]} \tag{2}$$

for all $x \in S, S \in 2^X \setminus \emptyset$ and $i, j \in \{1, 2\}$ with $j \neq i$.

[10] A stochastic choice rule p has a Luce representation (Luce, 1959), if there exists a weight distribution $w \in \Delta_{++}(X)$ such that $p(x, S) = \frac{w(x)}{\sum_{y \in S} w(y)}$ for all $x \in S, S \in 2^X \setminus \emptyset$. This ratio of relative weights is known as the "Luce ratio."

When (p_1, p_2) has a dual interaction representation with parameters $(w_1, w_2, \alpha_1, \alpha_2)$, we say that $(w_1, w_2, \alpha_1, \alpha_2)$ *represent* (p_1, p_2).

Dual interaction model defines an equilibrium, where the stochastic choice behavior of the agents ends up being contingent on each other in a particular way.[11] Different cognitive and/or interactive mechanisms may lead to the equilibrium granted by this model. For now, we abstract away from these underlying processes, and instead focus on identification and characterization.[12]

In our model, each p_i is only defined implicitly by the procedure in equation (2). p_2 needs to be known in order to determine p_1 and vice versa. However, given $(w_1, w_2, \alpha_1, \alpha_2)$, we can obtain an explicit representation by solving the system of simultaneous equations, arriving at:

$$p_i(x, S) \equiv \lambda_i(S)\frac{w_i(x)}{\sum\limits_{x\in S} w_i(x)} + (1 - \lambda_i(S))\frac{w_j(x)}{\sum\limits_{x\in S} w_j(x)} \tag{3}$$

for $\lambda_i(S) \in (0, 1)$ defined as,

$$\lambda_i(S) = \frac{w_i(S)[w_j(S) + \alpha_j]}{w_i(S)w_j(S) + \alpha_i w_j(S) + \alpha_j w_i(S)}$$

where $w_i(S)$ stands for $\sum\limits_{x\in S} w_i(x)$. Equation 3 helps to explain why we think of α_i as a measure of influence. The stochastic choice of i from choice set S is, geometrically, a convex combination of i's Luce choices and j's Luce choices. As α_i increases, this combination tends to be closer to j's Luce choices. In other words, the more the peer influence is, the higher is the weight attached to the peer's Luce ratio. In the extreme case, when $\alpha_i = 0$, $\lambda_i(S)$ is equal to 1, independent of the budget set, and the model

[11]It is also possible to think of individuals as if adjusting their behavior according to their *beliefs* about the behavior of their peer, rather than the behavior itself. Under an assumption of rational expectations, as it is common in social interactions literature (Blume et al., 2011), the beliefs happen to coincide with actual behavior. This interpretation is entirely in line with our model. However, since our main goal is to focus on the identification of underlying unobservable parameters out of the observable behavior, we choose not to include this additional dimension.

[12]Subsection 2.5 introduces several prominent examples to these underlying mechanisms.

boils down to standard Luce model.[13] In general, each p_i can be expressed as a linear combination of the Luce ratios, where, crucially, the weights in the combination depend on S. Observe this is "as if" each individual knows exactly not only her own intrinsic utilities but also those of the other individual, which are not necessarily observable. Notice that in our original formulation, each individual utilizes each others' observable choice behavior rather than their unobservable Luce weights. We believe influence based on observed behavior rather than an unobserved parameter is behaviorally and procedurally more plausible.[14]

Another important implication of this formulation is about uniqueness of the behavior produced, which is not obvious from the equilibrium description of the model. Since (p_1, p_2) can explicitly be expressed as functions of the preference parameters, for a given $(w_1, w_2, \alpha_1, \alpha_2)$, there is a unique pair (p_1, p_2) consistent with the dual interaction model. In other words, our model corresponds to a unique equilibrium.[15]

2.2. **Identification.** Assume we observe (p_1, p_2) that has a dual interaction representation. How can we identify the underlying preference and interaction parameters? A powerful feature of our model is that our identification strategy requires observation of behavior from only two menus: The universal set X and any menu S that has at least two distinct alternatives, say x and y. To see how, first define for each $i = 1, 2$,

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[13]It is worth noting that p_i consistent with dual interaction model does not satisfy IIA, the characterizing property of Luce model, ($\frac{p_i(x,S)}{p_i(y,S)} = \frac{p_i(x,T)}{p_i(y,T)}$ for all S, T and $x, y \in S \cap T$) in general; it only does so when $\alpha_i = 0$ or $\alpha_i \to \infty$. In the former, there is no influence, hence i behaves according to w_i, whereas in the latter, i fully mimics j. For an example to the violation of IIA by the dual interaction model, see the example given in the introduction: $\frac{p_{Dan}(home, \{home, walk\})}{p_{Dan}(walk, \{home, walk\})} = \frac{0.71}{0.29} \neq \frac{0.60}{0.26} = \frac{p_{Dan}(home, \{home, walk, gym\})}{p_{Dan}(walk, \{home, walk, gym\})}$.

[14]See Section 5 for further discussion on an alternative model that refers to a convex combination of Luce choices with set independent weights.

[15]Let us also note that although we restrict our attention to strictly positive stochastic choice rules (hence considered $w_i(\cdot) \in (0, 1)$), it is possible to extend the model to allow $w_i(\cdot) \in [0, 1]$. In this case two additional properties dealing with 0 probabilities are required for characterization of the model. Although this is a rather straightforward extension, the proof becomes tedious, hence we choose the restricted setting. The proof is available upon request.

for any pair (x, S) with $x \in S$, $d_i : (x, S) \mapsto \Re$, by

$$d_i(x, S) := p_i(x, S) - p_i(x, X).$$

The quantity $d_i(x, S)$ is simply the change in the probability of i's choosing x as the set X shrinks to S. With $\alpha_i \geq 0$, this change is always nonnegative, with the interpretation that in a larger set, there are more alternatives from which to choose.[16] In the dual interaction model, this change instead is governed by two separate effects. First, there is the individual effect. A larger set includes more alternatives, rendering any given alternative relatively less attractive. In addition, there is also a social influence effect imposed by the change of the other individual's choice probability, $d_j(x, S)$. With $\alpha_i > 0$, as the set enlarges, this indirect effect contributes to the loss in choice probability of any given alternative. Let us decompose $d_i(x, S)$ into these two effects explicitly for the model:

$$\begin{aligned}
d_i(x, S) =& p_i(x, S) - p_i(x, X) \\
=& \frac{1 - w_i(S)}{1 + \alpha_i} p_i(x, S) + \frac{w_i(S) + \alpha_i}{1 + \alpha_i} p_i(x, S) - \frac{1 + \alpha_i}{1 + \alpha_i} p_i(x, X) \\
=& \frac{1 - w_i(S)}{1 + \alpha_i} p_i(x, S) + \frac{w_i(x) + \alpha_i p_j(x, S)}{1 + \alpha_i} - \frac{w_i(x) + \alpha_i p_j(x, X)}{1 + \alpha_i} \\
=& \underbrace{\frac{1 - w_i(S)}{1 + \alpha_i} p_i(x, S)}_{\text{individual}} + \underbrace{\frac{\alpha_i}{1 + \alpha_i} d_j(x, S)}_{\text{social influence}}
\end{aligned}$$

The third line follows from the description of the model. Notice what is captured by the individual counterpart. In Luce's model, this loss is equal to $d(x, S) = \hat{p}(x, S) - \hat{p}(x, X) = \dfrac{w(x)}{w(S)} - w(x) = (1 - w(S))\hat{p}(x, S)$, where $\hat{p}(x, S)$ is the corresponding Luce probability. In our decomposition the individual counterpart captures a similar effect, but weighted by $1/(1 + \alpha_i)$.

We make use of this decomposition to infer α_i. One way of achieving this is to make use of a normalization and the decomposition of $d_i(y, S)$ to cancel out the individual counterparts. To this end, take an alternative $y \in S \setminus \{x\}$ and normalize both of

[16]Indeed this refers to the well-known Regularity property (Block and Marschak, 1960).

the decompositions by the respective observed probabilities as follows and take the difference:

$$
\begin{aligned}
\frac{d_i(x,S)}{p_i(x,S)} &= \frac{\frac{1-w_i(S)}{1+\alpha_i}p_i(x,S)}{p_i(x,S)} + \frac{\frac{\alpha_i}{1+\alpha_i}d_j(x,S)}{p_i(x,S)} \\
\frac{d_i(y,S)}{p_i(y,S)} &= \frac{\frac{1-w_i(S)}{1+\alpha_i}p_i(y,S)}{p_i(y,S)} + \frac{\frac{\alpha_i}{1+\alpha_i}d_j(y,S)}{p_i(y,S)} \\
\frac{d_i(x,S)}{p_i(x,S)} - \frac{d_i(y,S)}{p_i(y,S)} &= \frac{\alpha_i}{1+\alpha_i}\left[\frac{d_j(x,S)}{p_i(x,S)} - \frac{d_j(y,S)}{p_i(y,S)}\right] \qquad (4)
\end{aligned}
$$

Equation 4 reveals α_i uniquely whenever there exists (x,y,S) such that $\frac{d_j(x,S)}{p_i(x,S)} - \frac{d_j(y,S)}{p_i(y,S)} \neq 0$. Within the proof of Theorem 1, we show that there always exist (x,y,S) such that this condition holds, as long as $p_1 \neq p_2$. For the inference of $w_i(x)$, we simply make use of the description of the model for choices from X, yielding: $w_i(x) = p_i(x,X) + \alpha_i(p_i(x,X) - p_j(x,X))$. Obviously each $w_i(x)$ is identified uniquely with $\sum_X w_i(x) = 1$. Let us state these results in a proposition for completeness purposes.

Proposition 1. *Let $p_1 \neq p_2$ and (p_1,p_2) have a dual interaction representation. Then $(w_1,w_2,\alpha_1,\alpha_2)$ that represent (p_1,p_2) are identified uniquely.*

Identification above relies on the availability of data from two sets, the universal set X and any other menu S with at least two alternatives. This begs the question whether it is possible to do any inference when choices from X are not available? Indeed it is possible to recover the parameters from pairs of sets as long as they have at least two common elements, although the identification strategy gets slightly more complicated. To see how, let any two distinct sets S,T with $x,y \in S \cap T$ and $S \cup T = X$ and reproduce equation (4) for any two such S,T, as $d_i(x,S,T) = p_i(x,S) - p_i(x,T) =$

$$
\begin{aligned}
&= \frac{w_i(T)-w_i(S)}{w_i(T)+\alpha_i}p_i(x,S) + \frac{w_i(x)+\alpha_i p_j(x,S)}{w_i(T)+\alpha_i} - \frac{w_i(x)+\alpha_i p_j(x,T)}{w_i(T)+\alpha_i} \\
&= \underbrace{\frac{w_i(T)-w_i(S)}{w_i(T)+\alpha_i}p_i(x,S)}_{\text{individual}} + \underbrace{\frac{\alpha_i}{w_i(T)+\alpha_i}d_j(x,S,T)}_{\text{social influence}}
\end{aligned}
$$

Normalizing the decompositions for distinct $x, y \in S$ and taking the difference will result in:

$$\frac{d_i(x,S,T)}{p_i(x,S)} - \frac{d_i(y,S,T)}{p_i(y,S)} = \underbrace{\frac{\alpha_i}{w_i(T)+\alpha_i}}_{\gamma_i(x,y,S,T)} \left[\frac{d_j(x,S,T)}{p_i(x,S)} - \frac{d_j(y,S,T)}{p_i(y,S)}\right]$$

Thus two identifying equations are:

$$\gamma_i(x,y,S,T) = \frac{\alpha_i}{w_i(T)+\alpha_i} \text{ and } \gamma_i(x,y,T,S) = \frac{\alpha_i}{w_i(S)+\alpha_i}. \tag{5}$$

Unlike the case with data from X, we now have one too many parameters for unique identification only from γ_is. The third identity we need comes from the normalization assumption $w_i(X) = 1$. Yet as the behavior from X is not observed, we need to decompose it consistently over S and T. Since $w_i(S) + w_i(T \setminus S) = 1$, by definition of the model $w_i(x) = [\alpha_i + w_i(S)]p_i(x,S) - \alpha_i p_j(x,S)$ yields:

$$w_i(T \setminus S) = [\alpha_i + w_i(T)] \sum_{x \in T \setminus S} p_i(x,T) - \alpha_i \sum_{x \in T \setminus S} p_j(x,T) = 1 - w_i(S),$$

resulting in the last equation sufficient for unique identification combined with the two above.

We shall note that the requirement $S \cup T = X$ is not strictly necessary for identification without choice data from X. Since we cannot speculate about underlying parameters without observing some data involving all variables, the identification requires some observations covering X. Specifically, in addition to identification equations (5), more data revealing $w_i(S \cup T)$ is required. Whenever $S \cup T = X$, the normalization $w_i(X) = 1$ comes to aid. Whenever $S \cup T \neq X$, any additional observation revealing $w_i(X \setminus (S \cup T))$ should be sufficient for full identification. For instance, $p_i(z,R), p_i(t,Q)$ for $i = 1, 2$ will be sufficient if $z, t \in (R \cap Q)$ and $X \setminus (S \cup T) \subseteq (R \cup Q)$.

2.3. **Falsifiability.** For identification we assumed a pair of choice behaviors (p_1, p_2) consistent with the dual interaction model. We now need to express explicitly how one can detect the consistency of the data with the model. In other words for given (p_1, p_2), which properties of these behaviors ensure that these two individuals are behaving as if they are choosing according to dual interaction model?

We have three falsifiable characterizing properties built around the decomposition of $d_i(x,S)$ into individual and social counterparts as we have used in subsection 2.2. Specifically, for any $S \neq X$ and $x \in S$, $d_i(x,S)$ is composed of two counterparts: the individual effect (as there are more options in X than S for i's attraction) and the social influence effect (same goes for j's attraction).

Our characterizing properties build on the premise that one can eliminate the unobserved individual effects for $x \in S$ by cancelling them out with those of $d_i(y,S)$ for some distinct $y \in S$. The remainder will then be a function of the social influence effect. Specifically, it will be a linear function. Formally, take any S and $x, y \in S$ with $\dfrac{d_j(x,S)}{p_i(x,S)} - \dfrac{d_j(y,S)}{p_i(y,S)} \neq 0$ and define $\beta_i(x,y,S)$ as follows:

$$\frac{d_i(x,S)}{p_i(x,S)} - \frac{d_i(y,S)}{p_i(y,S)} = \beta_i(x,y,S)\left[\frac{d_j(x,S)}{p_i(x,S)} - \frac{d_j(y,S)}{p_i(y,S)}\right] \tag{6}$$

Three properties that impose conditions on these two variables $\beta_1(x,y,S)$ and $\beta_2(x,y,S)$ are sufficient for the characterization of the dual interaction model.

Independence [*I*]. *$\beta_i(x,y,S)(:= \beta_i)$ is independent of S, x, y. Moreover β_i satisfies (6) for all $S \neq X$ and distinct $x, y \in S$.*

Uniform Boundedness [*UB*]. *$\beta_i(x,y,S) < \min_{z \in X} \left\{\dfrac{p_i(z,X)}{p_j(z,X)}\right\}$ for all S, and distinct $x, y \in S$.*

Non-negativeness [*Nn*]. *$\beta_i(x,y,S) \geq 0$ for all S, and distinct $x, y \in S$.*

Independence is the property that restores the additive linear influence structure among individuals. $\beta_i(x,y,S)$ is defined for all those observations with a non-zero $\dfrac{d_j(x,S)}{p_i(x,S)} - \dfrac{d_j(y,S)}{p_i(y,S)}$. The first part of Independence ensures that $\beta_i(x,y,S)$ is indeed constant across observations, hence defining β_i. The second part of Independence guarantees that this β_i satifies equation (6) even for those observations with

$\dfrac{d_j(x,S)}{p_i(x,S)} - \dfrac{d_j(y,S)}{p_i(y,S)} = 0$. Uniform Boundedness guarantees that idiosyncratic evaluations of alternatives, w_i are positive. This is due to the choice of $\alpha_i := \dfrac{\beta_i}{1-\beta_i}$ and $w_i(x) := p_i(x,X) + \alpha_i(p_i(x,X) - p_j(x,X))$, as revealed in subsection 2.2. These two equations jointly imply: $\dfrac{p_i(x,X)}{p_j(x,X)} = \dfrac{w_i(x) + \alpha_i p_j(x,X)}{(1+\alpha_i)p_j(x,X)} = \dfrac{w_i(x)}{(1+\alpha_i)p_j(x,X)} + \dfrac{\alpha_i}{1+\alpha_i}$. Hence, by UB, $\beta_i = \dfrac{\alpha_i}{1+\alpha_i} < \min_{z \in X} \left\{ \dfrac{w_i(z)}{(1+\alpha_i)p_j(z,X)} + \dfrac{\alpha_i}{1+\alpha_i} \right\}$ ensures that $w_i(z) > 0$ for all z. And finally, Non-negativeness restricts the interaction among individuals to conformity behavior rather than diversification.

The characterization result is stated for pairs of stochastic choice rules with some variation in the overall behavior, i.e., $p_1 \neq p_2$. This is because having exactly the same behavior in any choice set might be due to identical preferences of 1 and 2, i.e, $w_1 = w_2$; or it might be because one of the individuals only cares about imitating the other individual. It is not possible to distinguish between these cases without any additional information, such as their choice behavior in isolation.

Theorem 1. *Let $p_1 \neq p_2$. Then (p_1, p_2) has a* ***dual interaction*** *representation if and only if it satisfies Independence, Uniform Boundedness, and Non-negativeness.*

The proof constructs the model thanks to the structure granted by Independence and by the help of restrictions imposed by the remaining two axioms. We take $\alpha_i(x,y,S) := \alpha_i = \dfrac{\beta_i}{1-\beta_i}$ (well-defined by the first two properties and non-negative by the latter two) and $w_i(x) := p_i(x,X) + \alpha_i(p_i(x,X) - p_j(x,X))$ (positive by Uniform Boundedness). We then show that for any S and $x, y \in S$, Independence builds up to

$$\frac{p_i(x,S)}{p_i(y,S)} = \frac{w_i(x) + \alpha_i p_j(x,S)}{w_i(y) + \alpha_i p_j(y,S)}.$$

The fact that this holds for each pair of alternatives immediately gives us the dual interaction model.

Theorem 1 is a strong result. Three properties over $\beta_i(\cdot)$ are necessary and sufficient to confirm if two individuals are choosing consistently with the dual interaction model. This becomes a straightforward falsification exercise for an observable pair of choice behaviors, (p_1, p_2), as $\beta_i(\cdot)$ is merely derived from data. Independence is a property very much in the spirit of 'constant ratio' properties such as Luce's IIA. IIA requires that the ratio of choice frequencies of any two alternatives is constant across sets. Similarly, Independence requires that the ratio given by $\beta_i(\cdot)$ for any two alternatives is constant across sets. Certainly what is captured by β_i is not as straightforward to see as Luce's ratio, however we argue that there is subtle behavioral content to β_i. Observe that, $\dfrac{d_i(x,S)}{p_i(x,S)}$ is the percentage decrease in agent i's choice probability of x in expanding S to X. So, $\dfrac{d_i(x,S)}{p_i(x,S)} - \dfrac{d_i(y,S)}{p_i(y,S)}$ is a differential in percentage changes. On the other hand, $\dfrac{d_j(x,S)}{p_i(x,S)} - \dfrac{d_j(y,S)}{p_i(y,S)}$ reflects a differential in percentage changes for agent j, normalized by the choice probabilities of i. Thus, the ratio of these two differentials in percentage changes, i.e., $\beta_i(x,y,S)$, acts like a differential cross-elasticity of choice probabilities in expanding the set S to X. Independence fixes this differential cross-elasticity for different menus, while the other two properties bound it.

2.4. **Stability.** The dual interaction model involves an adjustment procedure where an individual's evaluation of an alternative is adjusted by the other's behavior as well as the level of susceptibility to influence. We now embed this adjustment procedure in a dynamic setting, where individuals start interaction from possibly unrelated behaviors. Specifically let (p_1^t, p_2^t) denote the behaviors of 1 and 2 at period $t > 0$ and assume that their initial behaviors (p_1^1, p_2^1) are given. One can think of new roommates or teenagers just enrolled in a new school as examples. Below we show that although these individuals start interacting from possibly unrelated behaviors, as long as they adjust consistently, eventually they converge to (p_1^*, p_2^*), the unique pair of behaviors that the model yields for the given set of parameters. In other words, the behavior produced by the dual interaction model constitutes a stable equilibrium when embedded in a dynamic environment.

Theorem 2. *Take $w_i \in \Delta_{++}(X)$, $\alpha_i \geq 0$, $p_i^*(S) \in \Delta_{++}(S)$ for all $S \in 2^X \setminus \{\varnothing\}$ and for each $i \in \{1,2\}$ and let $(w_1, w_2, \alpha_1, \alpha_2)$ represent (p_1^*, p_2^*). Further, let $(p_1^1, p_2^1) \in$*

$\Delta(S) \times \Delta(S)$. *Define for each* $i \in \{1,2\}$ *and* $t \geq 2$, $p_i^t(\cdot, S) \in \Delta(S)$ *via*

$$p_i^t(x,S) \equiv \frac{w_i(x) + \alpha_i p_j^{t-1}(x,S)}{\sum_{y \in S} w_i(y) + \alpha_i p_j^{t-1}(y,S)}.$$

Then for each $i \in \{1,2\}$, $\lim_{t \to \infty} p_i^t = p_i^*$.

An interesting implication of this dynamic environment involves identification. Although the observed behavior changes over time, because it changes in a consistent way, our identification strategy still holds for the underlying preference and interaction parameters $(w_1, w_2, \alpha_1, \alpha_2)$. Similar to the static setting, the data requirement is minimal: only choice behavior from two different sets need be observed. However, since now observations are from different time periods, inference of α_i demands data from two successive periods. To see how, let us reproduce equation (4) for this dynamic environement. Take any $S \neq X$ with $x, y \in S$ such that $\dfrac{d_j^{t-1}(x,S)}{p_i^t(x,S)} - \dfrac{d_j^{t-1}(y,S)}{p_i^t(y,S)} \neq 0$ and let:

$$\beta_i(x,y,S) = \frac{\dfrac{d_i^t(x,S)}{p_i^t(x,S)} - \dfrac{d_i^t(y,S)}{p_i^t(y,S)}}{\dfrac{d_j^{t-1}(x,S)}{p_i^t(x,S)} - \dfrac{d_j^{t-1}(y,S)}{p_i^t(y,S)}}.$$

$$\begin{aligned}
\text{Then, } d_i^t(x,S) =& p_i^t(x,S) - p_i^t(x,X) \\
=& \frac{1 - w_i(S)}{1+\alpha_i} p_i^t(x,S) + \frac{w_i(S) + \alpha_i}{1+\alpha_i} p_i^t(x,S) - \frac{1+\alpha_i}{1+\alpha_i} p_i^t(x,X) \\
=& \frac{1 - w_i(S)}{1+\alpha_i} p_i^t(x,S) + \frac{w_i(x) + \alpha_i p_j^{t-1}(x,S)}{1+\alpha_i} - \frac{w_i(x) + \alpha_i p_j^{t-1}(x,X)}{1+\alpha_i} \\
=& \underbrace{\frac{1 - w_i(S)}{1+\alpha_i} p_i^t(x,S)}_{\text{individual}} + \underbrace{\frac{\alpha_i}{1+\alpha_i} d_j^{t-1}(x,S)}_{\text{social influence}}
\end{aligned}$$

Then, the difference between normalized decompositions for distinct $x, y \in S$ yields:

$$\frac{d_i^t(x,S)}{p_i^t(x,S)} - \frac{d_i^t(y,S)}{p_i^t(y,S)} = \frac{\alpha_i}{1+\alpha_i} \left[\frac{d_j^{t-1}(x,S)}{p_i^t(x,S)} - \frac{d_j^{t-1}(y,S)}{p_i^t(y,S)} \right]$$

Hence, we have

$$\beta_i(x, y, S) = \frac{\alpha_i}{1 + \alpha_i}$$

as before. Identification of $w_i(x)$ is achieved via:

$$w_i(x) = (1 + \alpha_i)p_i^t(x, X) - \alpha_i p_j^{t-1}(x, X).$$

We conclude the analysis of our baseline model by stating this identification result.

Proposition 2. *Let* $(p_1^{t-1}, p_2^{t-1}, p_1^t, p_2^t)$ *such that for each* $i \in \{1, 2\}$ *and* $p_i^t(\cdot, S) \in \Delta(S)$

$$p_i^t(x, S) \equiv \frac{w_i(x) + \alpha_i p_j^{t-1}(x, S)}{\sum_{y \in S} w_i(y) + \alpha_i p_j^{t-1}(y, S)}.$$

Then $(w_1, w_2, \alpha_1, \alpha_2)$ *that represent* (p_1, p_2) *are identified uniquely.*

2.5. **Foundations.** Why does the dual interaction model make sense as a decision procedure that incorporates social influence? We provide three different foundational justifications, three different mechanisms that produce behavior consistent with the dual interaction model. Each environment differs from the classical stochastic choice setting. To this end, we strip the menu-richness of the choice argument away and focus on a single budget set, say X. We suppress the menu dependence in the notation of this subsection. All of the following can be reproduced for any menu S.

The first mechanism we introduce reproduces dual interaction as the equilibrium of a game, whereas the second one incorporates individual utility maximization in a discrete choice setting with peer effects. The main link between these two and our model is built around the use of a logistic distribution. However as we show in the third mechanism, the logistic set-up is dispensable. This last part introduces a simple naive learning mechanism that also reproduces dual interaction behavior in the limit.

2.5.1. *Game theoretic foundations:* Dual interaction model envisions individual behavior contingent on peer behavior, which naturally relates to a game set-up. Thus the first question we investigate is whether the pair of behaviors produced by the dual interaction model could also be rationalized by an underlying game. Indeed we show that, a very specific solution concept for normal form games, Quantal Response Equilibrium (McKelvey and Palfrey, 1995), also reproduces the behavior granted by our model. To see this, consider a normal form game with two players 1 and 2, with $S = S_1 \times S_2 = X \times X$ as the set of strategy profiles and s_i represents a pure strategy for player i. Let Σ_i denote the set of probability distributions over S_i and an element $\sigma_i \in \Sigma_i$ is a mixed strategy, and $\sigma_i(s_i)$ is the probability that player i chooses pure strategy s_i with Σ as the set of mixed strategy profiles. The pay-off functions $u_i : S \to \Re$ are such that $u_i(x, y)$ represents the utility of player i when player 1 consumes x and player 2 consumes y. In particular, assume that $u_1(s) = u_1(x, y) = w_1(x) + \alpha_1 \mathbf{1}\{x = y\}$ and $u_2(s) = u_2(x, y) = w_2(y) + \alpha_2 \mathbf{1}\{y = x\}$. In other words each player receives a consumption utility $w_i(x)$ and additional utility α_i when their consumptions match. With positive α, this corresponds to a very simple form of pay-off function for conformity games. For instance, consider classroom behavior of students: Asking a question 'feels easier' when someone else does so (Alessio and Kilgour, 2011) or negative behavior such as aggression becomes more rewarding in presence of aggressive peers (Hanish et al., 2005).

Hence, for each mixed-strategy profile $\sigma \in \Sigma$, player i's expected payoff is $u_i(\sigma) = \sum_{s \in S} \sigma_i(s_i)\sigma_j(s_j)u_i(s)$ and the expected payoff for adopting the pure strategy s_i when the other player uses σ_j is $u_i(s_i, \sigma_j) = \sum_{s_j \in S_j} \sigma_j(s_j)u_i(s_i, s_j) = \sigma_j(s_i)(w_i(s_i) + \alpha_i) + (1 - \sigma_j(s_i))w_i(s_i) = w_i(s_i) + \alpha_i\sigma_j(s_i)$. Under the assumption that $U_i(s_i, \sigma_j) = u_i(s_i, \sigma_j)\varepsilon_{is}$ with i.i.d. Log-logistic errors (i.e., $\log \varepsilon_i$ follows a Type 1 extreme value distribution), the QRE outcome coincides with (p_1, p_2) of the dual interaction model. The stochastic derivation is provided in the Appendix.

Two caveats must be mentioned: first, QRE is a prediction for a single game, whereas the testable implications of our model derive their power from the ability to observe behavior *across* choice sets. Indeed, QRE affords basically no predictions on a single-game (much like classical choice theory generates no predictions from a single budget). See for example, Haile et al. (2008). Thus, a suitable extension of the notion of QRE *across* game forms must be described.[17] Second, our model results from a very specific choice of error distribution (one of the parameters of the QRE model) and a very specific choice of utility (the other main parameter). To sum up, the behavior produced by our model may be viewed as being *rationalized* by a particular choice of game forms and the logit QRE solution concept, suitably extended to across games. We believe exploration of similar results for generic games of peer influence with standard equilibrium concepts remains as an interesting open question outside the scope of this paper.[18]

2.5.2. *Random utility with linear social interactions:* The standard econometric tools to study social interactions include discrete choice models with peer effects (Blume, 1993; Brock and Durlauf, 2001, 2006). These models regard individual utility as a linear additive function of observed and unobserved individual characteristics as well as social influence. Under the assumption of i.i.d extreme value unobserved characteristics, utility maximization yields choice frequencies as a function of individual characteristics and social influence. The dual interaction model can also be reproduced in a multinomial discrete choice setting. Two specific assumptions are sufficient to achieve this: a logarithmic transformation of the utility and a relevant extreme value distribution. To see how this works, assume a multiplicative form for individual utility as follows:

$$U_i(x) = V_i(x)\varepsilon_i(x) \quad \text{where } V_i(x) = w_i(x) + \alpha_i p_j(x)$$

Similar to the previous subsection (and as by step by step derivation provided in the Appendix), under the assumption that ε_i follows a log-logistic distribution, maximization of $\log U_i(x)$ results in $p_i(x)$, exactly as given by the dual interaction model. Thus, a logarithmic transformation of the individual utility and a relevant extreme value distribution for the error terms in a discrete choice setting with social interactions lead to the behavior described by the dual interaction model.

[17] In particular one must take care to ensure the error distributions across game forms coincide in a natural way.

[18] The extensive literature on peer influence games over social networks (Ballester et al., 2006; Calvo-Armengol et al., 2009) does not provide an immediate answer to this question, mostly because the multivariate discrete nature of our setting and the assumption that main observables are stochastic choice outcomes over different menus.

According to Blume et al. (2011) empirical challenges to identification of social interactions are broadly grouped under three categories: (i) simultaneous equations problem: how to differentiate the direct interdependencies between choices from the effects of predetermined social factors; (ii) unobserved group-level characteristics; (iii) endogeneity of reference groups and self-selection. The primary aim of our social interaction model for identification purposes is the revelation of the direct interdependencies between choices. Those are captured by the interaction parameter, α_i. Since our model lives in a two-parameter world, all other effects are left to be captured by the preference parameter w_i. This approach enables us to tackle the simultaneous equation problem of (i), 'the reflection problem', by identifying the endogenous effects and abstracting away from the contextual effects as well as the other unobservables.

In order to address challenges belonging to (ii)[19], one strategy could be to introduce group level unobservables at the random utility stage, and see how this affects the stochastic derivation. Specifically, let

$$U_i(x) = V_i(x)e_i(x) \quad \text{where } V_i(x) = w_i(x) + \alpha_i p_j(x) \ \text{ and } e_i(x) = \varepsilon_i(x)\mu(x)$$

where e_i follows a log-logistic distribution as before.[20] It is immediate to see that the stochastic derivation would result in dual interaction model with the same parameters. In other words, group level unobservables in the utility function become idle for the choice behavior as long as the joint distribution of individual level and group level disturbances comply with multinomial log-logit derivation.

Another approach to address the challenges belonging to (ii) or (iii) or to investigate the effects of predetermined social factors would be to further explore heterogeneities over w_i (and/or α_i). For instance, take the issue of homophily, the tendency to create social ties with people who are similar to one's self (McPherson et al., 2001; Blackwell

[19]See Brock and Durlauf (2007) for identification of correlated effects in discrete choice social interaction models and Bramoullé et al. (2020) for a recent survey of the methods developed to address it.

[20]Hence $\log \varepsilon_i(x) + \log \mu(x)$ follows a standard Gumbel distribution. That such a decomopsition exists follows from the work of Shanbhag and Sreehari (1977) (see e.g. equation (2) there). Bosch and Simon (2013) offers another application. Admittedly, the choice of this joint distribution is ad hoc, however any other convolution would potentially result in choice probabilities that are entirely different to our model.

and Lichter, 2004; Currarini et al., 2009). This is an endogenous reference group formation problem and is not immediate to identify out of observable behavior. However our model reduces homophily to the similarity of underlying w_i parameters for people with high α_i values. In other words, our identification strategy can be helpful to identify homophilic interactions by comparing the revealed w_i's. Certainly this becomes a more interesting question in a multi individual setting, as we explore in Section 3.

One final potential challenge that may arise in our setting but not listed explicitly within the above categories is due to the exogenous menu variation across individuals. We assume that individuals choose from the same menus of alternatives, and our entire identification strategy is based on menu variation. However in cases the menus available to individuals are correlated with the idiosyncratic unobservables, this critical assumption fails. Hence the dual interaction representation will not be useful for identification with endogenous menu variation.

2.5.3. *Naive learning with anchors:* The previous two subsections have explored the rational and/or strategic motivations underlying dual interaction mechanism. However adopting the behavior dictated by the dual interaction model does not necessitate adopting standard notions of full rationality. Indeed, as we now show, dual interaction model can also be reproduced in a particular boundedly rational learning setting. The most well-known model of naive learning over social networks, the DeGroot model, envisions a non-Bayesian updating of individual beliefs by repeatedly taking weighted averages of one's neighbors' beliefs (DeGroot, 1974; Golub and Jackson, 2010).

In a DeGroot setting, each agent $n \in N$ has a belief $p_i(t) \in [0,1]$ at time $t \in \{0,1,2,...\}$. These beliefs might be thought as the probability that a statement is true, the likelihood of choosing an action or a measure of the quality of a given product, etc. Given the stochastic interaction matrix $T_{n\times n}$, where T_{ij} captures the influence of agent j on i, the updating rule is simply $p(t) = Tp(t-1) = T^t p(0)$, where $p(\cdot)$ stands for the vector of beliefs of all agents. Simply put, at each point in time, the individuals update their beliefs by taking a weighted average of their peers' and their own previous beliefs, with time invariant weights.

DeGroot model is essentially a belief updating model, where dual interaction model can be seen as a behavior adjustment model. In the following, we stick to our terminology and interpret p_i as stochastic behavior, rather then beliefs. However, as noted in footnote 11, the mathematics of our model is entirely consistent with a belief-based interpretation. Now, to see the relationship to our model let $N = 2$ -although the extension to the n individual case is immediate. Let,

$$p(0) = \begin{pmatrix} w_1(x) & w_2(x) & p_1^0(x) & p_2^0(x) \end{pmatrix}'$$

for some alternative $x \in X$, where $p_i^0 \in [0, 1]$ is any initial behavior, which might or might not be the same with the anchor, w_i.[21] In this setting, the anchors w_1 and w_2 can be seen as innate preferences/beliefs that do not change over time. Note that we only focus on one alternative, x, to keep things simple. The same can be done for all alternatives in the menu. Consider the following transition matrix T:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \dfrac{1}{1+\alpha_1} & 0 & 0 & \dfrac{\alpha_1}{1+\alpha_1} \\ 0 & \dfrac{1}{1+\alpha_2} & \dfrac{\alpha_2}{1+\alpha_2} & 0 \end{pmatrix}.$$

The first two rows indicate the time-independency of the anchors, whereas the last two rows correspond to the updating weights of the agents; for instance, individual 1 is influenced by individual 2 with a weight of $\dfrac{\alpha_1}{1+\alpha_1}$, whereas her own anchor has a weight of $\dfrac{1}{1+\alpha_1}$ and so on. As before, the interaction paramater α_i captures the relative importance of social interaction effects. With this transition matrix, period 1 behavior will be

$$p(1) = Tp(0) = p(1) = \begin{pmatrix} w_1(x) & w_2(x) & \dfrac{w_1(x) + \alpha_1 p_2^0(x)}{1+\alpha_1} & \dfrac{w_2(x) + \alpha_2 p_1^0(x)}{1+\alpha_2} \end{pmatrix}.$$

[21] Friedkin and Johnsen (1990) suggest a generalization of the DeGroot model where updating at each period also involves agents' initial beliefs. They also show convergence to a non-consensus state. The slight difference with the dynamic version of our model, as we examine in subsection 2.4 is that, for their model the initial behavior is equal to the initial belief. Instead we show convergence to the behavior dictated by our model for any p_0.

As we also prove in subsection 2.4, in the limit $p(t) = T^t p(0)$ indeed converges to a $\begin{pmatrix} w_1(x) & w_2(x) & p_1^*(x) & p_2^*(x) \end{pmatrix}$ with $(p_1^*(x), p_2^*(x))$ as defined by the dual interaction model. In a nutshell, dual interaction model can also be reproduced as the limit of a DeGroot updating process with anchors.

Overall, these three settings indicate that behavior postulated by our model can be justified by an underlying utility maximization as well as a naive learning mechanism. The main difference of our model lies in the menu variability of our setting. Our model is a stochastic choice model that assumes consistent behavior across menus. Critically this menu variability grants us unique identification of the underlying unobserved parameters.

3. Multi-agent Interaction

One of the strengths of our model is that it is easily generalizable to multi individual settings with more intricate forms of social interactions. We can easily capture the heterogeneities that drive different behavioral outcomes in a social context. Not only individuals have different preferences but they also have different levels of susceptibility to influence. Or similarly, different people might influence an individual in different ways. The generalization of our model to multi individual settings allow for these variations, by providing a complementary approach to the identification of social interactions over social networks. In particular, it allows the identification of a weighted social network from choice behavior.

Early works on social networks have assumed known network structure, based on common observables or self-reported, elicited data (Bramoullé et al., 2009; Lee et al., 2010; De Giorgi et al., 2010), that is rather costly to collect (De Paula, 2017). A first improvement on this was suggested by Blume et al. (2015) by assuming only partial information on the structure of the underlying network. De Paula et al. (2019) advances on this by assuming no a priori information on the network structure and provides sufficient conditions for full identification of social interactions with panel data. Similarly, our generalized model do not require any exogenous network structure. On the contrary, our representation theorem reveals the unknown network of social influence in addition to individual preferences and influence patterns. Specifically, given the behavior of a group of individuals that is consistent with our characterizing properties, we can *uniquely* identify the underlying preferences, represented by w_i, and the interaction patterns, represented by α_{ij}, capturing how individual i is influenced by the behavior of individual j for all pairs of individuals i and j. Note that the interaction between i and j might be asymmetric, i.e., α_{ij} need not be equal to α_{ji}.

Let us now formally introduce the multi individual model. Let N denote a set of $n < +\infty$ individuals interacting. As before, for each choice problem, $S \in 2^X \setminus \emptyset$, we observe agent i's stochastic choice, $p_i(x,S)$. Let $\boldsymbol{p}_{-i}(x,S) \in \Re^{n-1}$ denote the vector of $p_j(x,S)$ and $\boldsymbol{d}_{-i}(x,S) \in \Re^{n-1}$ the vector of $d_j(x,S)$ for all $j \neq i$.

Definition. $(p_1, p_2, ..., p_n)$ has a **social interaction** representation if for each $i \in N$ there exist $w_i \in \Delta_{++}(X)$ and $\boldsymbol{\alpha}_i \in \Re^{n-1}_+$ such that

$$p_i(x,S) = \frac{w_i(x) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(x,S)}{\sum_{y \in S}[w_i(y) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(y,S)]}$$

for all $x \in S$ and for all S.

The parameter $\boldsymbol{\alpha}_i$ captures different levels of susceptibility to influence from different individuals, i.e., agent i can be influenced differently by different j's. Let α_{ij} denote the entry of $\boldsymbol{\alpha}_i$ relating to the influence of individual j on i. If $\alpha_{ij} = 0$ for all $j \neq i$, once again i's choice behavior reduces down to Luce.

The identification strategy and the characterizing properties are similar to those of the baseline model. Notice that for any $S \neq X$, and any two distinct $x, y \in S$, now there might be multiple vectors $\boldsymbol{\gamma}_i \in \Re^{n-1}$ satisfying the following equation:

$$(7) \qquad \boldsymbol{\gamma}_i \cdot \left(\frac{\boldsymbol{d}_{-i}(x,S)}{p_i(x,S)} - \frac{\boldsymbol{d}_{-i}(y,S)}{p_i(y,S)} \right) = \frac{d_i(x,S)}{p_i(x,S)} - \frac{d_i(y,S)}{p_i(y,S)}.$$

We will be interested in the ones that satisfy it for all observations.

$$\mathcal{B}_i = \{\boldsymbol{\gamma}_i \in \Re^{n-1} \,|\boldsymbol{\gamma}_i \text{ solves (7) for any } S \text{ and distinct } x, y \in S\}$$

The first characterizing property ensures that $\mathcal{B}_i$ is nonempty, hence there is at least one solution to the system of equations given by (7) for all S and $x, y \in S$. The last one puts bounds on it.

N-Independence [*N-I*]. *$\mathcal{B}_i$ is nonempty.*

N-Positive Uniform Boundedness. [***N-PUB***] *For all $z \in X$, $p_i(z, X) > \boldsymbol{\gamma}_i \cdot \boldsymbol{p}_{-i}(z, X)$ for some $\boldsymbol{\gamma}_i \in \mathcal{B}_i$ with $\boldsymbol{\gamma}_i \in \Re_+^{n-1}$.*

N-Independence implies that there exists a vector, say $\boldsymbol{\beta}_i$, that satisfies (7) independent of S, x, y. As before, $\boldsymbol{\alpha}_i$ is to be identified from $\boldsymbol{\beta}_i$. Specifically, $\alpha_{ij} = \dfrac{\beta_{ij}}{1 - \sum\limits_{j \neq i} \beta_{ij}}$. However, unique identification requires more than two observations this time, simply because there are more unknowns now. Indeed, equation (7) has $(n-1)$ unknowns, α_{ij} for each $j \neq i$. Hence, the number of linearly independent equations required to solve the system is $(n-1)$. Notice that this does not mean we necessarily need data from $(n-1)$ different menus. All that is required is $(n-1)$ observations; data from two different menus is sufficient as long as there are at least $(n-1)$ common pairs of alternatives in these two menus.[22]

Unique identification of the underlying preferences is then achieved via

$$w_i(x) = p_i(x, X) + \sum_{j \neq i} \alpha_{ij} [p_i(x, X) - p_j(x, X)]. \tag{8}$$

Theorem 3. *Let $\{p_i\}_{i \in N}$. Then, $\{p_i\}_{i \in N}$ has a* ***social interaction*** *representation if and only if N-Independence and N-Positive Uniform Boundedness hold.*

[22] Notice that with n individuals, there are $n(n-1)$ unknown interaction parameters. Full identification of these $n(n-1)$ unknowns for our model requires $(n-1)$ independent identification equations given by equation 7, which corresponds to observations of $(n-1)$ *pairs* of alternatives from at least 2 different menus. For instance with 4 individuals, to point identify 12 interaction parameters, observations from $p_i(\{x, y, z\})$ and $p_i(X)$ with $|X| \geq 4$ is sufficient (conditional on linear independence). When number of alternatives in X is not high enough to consider different pairs, it is possible to use the same pairs of alternatives from a larger number of menus. With 10 individuals, to identify 90 interaction parameters, observations from $p_i(\{x, y, z, t, u\}$ and $p_i(X)$ with $|X| \geq 6$ is sufficient as well as $p_i(\{x, y, z\})$, $p_i(\{x, y, t\})$, $p_i(\{x, z, t\})$ and $p_i(X)$ with $|X| \geq 4$.

As before, the equilibrium defined by the model always exists and is unique. Moreover, when embedded in a dynamic adjustment process, as in subsection 2.4, the limit behavior happens to be the equilibrium defined by our model. The following theorem formalizes these.

Theorem 4. *Take* $w_i \in \Delta_{++}(X), \alpha_{ij} \geq 0$ *for all* $i, j \in \{1, 2, ..., n\}$ *with* $i \neq j$. *Then, there is a unique* $(p_1^*, \dots, p_N^*) \in \Delta_{++}(S)^N$ *for which*

$$p_i^*(x, S) = \frac{w_i(x) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}^*(x, S)}{\sum_{y \in S}[w_i(y) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}^*(y, S)]}$$

and for any $(p_1^1(\cdot, S), \dots, p_N^1(\cdot, S)) \in \Delta_{++}(S)^N$, *the iterative map*

$$p_i^t(x, S) = \frac{w_i(x) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}^{t-1}(x, S)}{\sum_{y \in S}[w_i(y) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}^{t-1}(y, S)]}$$

converges to $(p_1^*, \dots, p_N^*)$.

4. Negative Interactions

Most of the theoretical tools developed to study social interactions are restricted by strategic complementarity or conformity type assumptions. This is because they only focus on positive interactions, where the individual payoff of an action increases the more it is chosen by one's peers. However in certain contexts, where individuals especially do not want to behave similarly, negative interactions are in play. An intuitive example to this is fashions and fads. A trend setter happens to be the one that initially behaves differently than everyone else. The choice of a fashion product not only signals which social group you would like to identify with but also signals who you would like to differentiate from (Pesendorfer, 1995). Among criminals competition for resources governs the need for negative interactions (Glaeser et al., 1996). Bhatia and Wang (2011) study peer effects in physicians' prescription behavior and find significantly negative peer influence, partly explained by observational learning and congestion effects. Foster and Rosenzweig (1995) find evidence of negative relation between experimental technology adoption rates of farmers and their neighbors. Other examples to settings with negative interactions include market entry games (Rapoport et al., 2000; Duffy and Hopkins, 2005) as well as anti-coordination games (Bramoullé, 2007; Bramoullé et al., 2004) and games that bring both coordination and anti-coordination motives together such as fashion games (Cao et al., 2013).[23]

The versatility of the dual interaction model allows us to extend it to capture negative interactions in a rather straightforward way. However we shall first point out that this is not as simple as taking any negative α_i. Let us explain: Consider our benchmark model, with two individuals i and j, and a pair (x, S) with $x \in S$. We refer to a *negative interaction* between i and j as the following phenomenon: Whenever j increases their propensity to choose x from S, i decreases her propensity in response. Formally, imagine two hypothetical behaviors from individual j, say $p_j(x, S)$ and $q_j(x, S)$, where $p_j(x, S) > q_j(x, S)$. *Negative interaction* refers to the property that if $p_i(x, S) = \dfrac{w_i(x) + \alpha_i p_j(x, S)}{w_i(S) + \alpha_i}$ and $q_i(x, S) = \dfrac{w_i(x) + \alpha_i q_j(x, S)}{w_i(S) + \alpha_i}$, then $p_i(x, S) < q_i(x, S)$.

4.1. **Reparametrization:** First, notice that a negative α_i does not necessarily imply negative interaction. Crucially, for values of $\alpha_i < 0$, whenever $w_i(S) < |\alpha_i|, p_i(x, S)$ puts a negative weight on $w_i(x)$ and a positive weight on $p_j(x, S)$, quite contrary to the essence of negative interactions. Thus, we employ a simple reparametrization of the model in order to avoid confusion. For the two-agent model, let $\delta_i \equiv \dfrac{1}{1 + \alpha_i}$, and observe that

$$p_i(x, S) = \frac{\delta_i w_i(x) + (1 - \delta_i) p_j(x, S)}{\delta_i w_i(S) + 1 - \delta_i}.$$

We maintain the premise that w_i remains a "weight of choice" absent any influence, so we hypothesize that $\delta_i > 0$.[24] Observe that the case of $\alpha_i \geq 0$ corresponds to $\delta_i \leq 1$ and for values of $\delta_i > 1$, $p_i(x, S)$ is indeed decreasing in $p_j(x, S)$ as required by the notion of negative interaction. Hence, with two agents, the first parametric restriction

[23]In network literature, negative ties are mostly interpreted as the conceptualization of dislike, opposition, antagonism and avoidance. For instance Bonacich and Lloyd (2004) investigate the effects of negative ties on status formation where where being disliked by popular individuals deters status. Everett and Borgatti (2014) examine how standard measures could be extended to networks including negative ties. Kaur and Singh (2016) survey the literature on online social networks that include negative ties. Our interpretation of negative influence is different than this literature, mainly because we focus on the influence from observed behavior.

[24]Notice that a negative δ_i implies a positive denominator for $p_i(x, S)$, deeming the weight of $w_i(x)$ negative and the weight of $p_j(x, S)$ positive, contradicting the notion of negative interaction defined above.

required to have a meaningful interpretation of negative influence as defined above is $\delta_i > 0$. The second restriction stems from the fact that p_i is a probability. We require $\delta_i w_i(x) + 1 - \delta_i \geq 0$ for each x. The detailed explanation for this restriction is left to the appendix.

4.2. **Generalization to multi-agent setting:** We now introduce our most general model of social interactions via a reparametrization of the previously introduced social interaction model and parametric bounds that ensure stability and existence. To this end, recall the model introduced in section 3, whereby for each $i, j \in N$ with $i \neq j$, α_i^j is the influence that j exerts on i. Let $\delta_i \equiv \dfrac{1}{1 + \sum_{j \neq i} \alpha_{ij}}$, $\delta_{ij} \equiv \dfrac{\alpha_{ij}}{1 + \sum_{j \neq i} \alpha_{ij}}$ where $1 + \displaystyle\sum_{j \neq i} \alpha_{ij} > 0$. Observe that $\delta_i + \displaystyle\sum_{j \neq i} \delta_{ij} = 1$.

Definition. $(p_1, p_2, ..., p_n)$ has a **general social interaction** representation if for each $i \in N$ there exist $w_i \in \Delta_{++}(X), \delta_i > 0$, and $\boldsymbol{\delta}_i \in \Re^{n-1}$ such that

(1) $\delta_i + \displaystyle\sum_{j \neq i} \delta_{ij} = 1$

(2) For every x, $\delta_i w_i(x) + \displaystyle\sum_{j \neq i} \min\{0, \delta_{ij}\} > 0$

and

$$p_i(x, S) = \frac{\delta_i w_i(x) + \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(x, S)}{\delta_i w_i(S) + \boldsymbol{\delta}_i \cdot \mathbf{1}}$$

for all S and all $x \in S$.

Similar to the two agent case, $\delta_i > 0$ is necessary to ensure a meaningful representation with negative influence. This is the first requirement for the validity of the general model. Condition (2) ensures that every profile of probability measures of the other agents is mapped to one with full support, as we explain in details in the appendix. As we establish in Theorem 6, it provides the convergence of the dynamic adjustment process. Once again, the detailed explanation for this choice of parametric bounds is left to the appendix.

4.3. **Identification, Falsifiability and Stability:** The identification strategy and the characterization of the general model is very similar to that of the social interaction model. The identification equation, equation (7) remains the same, hence *N-Independence* functions as the main characterizing property. Since the main difference between these two models is the set of admissible values for the interaction coefficients, a general boundedness property, that takes care of the bounds on the revealed $\boldsymbol{\gamma}_i$ is required.

GN-Uniform Boundedness. [***GN-UB***] *For all* $z \in X$, $p_i(z, X) > \boldsymbol{\gamma}_i \cdot \boldsymbol{p}_{-i}(z, X) - \sum_{j \neq i} min\{0, \gamma_{ij}\}$ *for some* $\boldsymbol{\gamma}_i \in \mathcal{B}_i$ *with* $\sum \gamma_{ij} < 1$.

Theorem 5. *Let* $\{p_i\}_{i \in N}$. *Then,* $\{p_i\}_{i \in N}$ *has a **general social interaction** representation if and only if N-Independence and GN-Uniform Boundedness hold.*

Unique identification of $\boldsymbol{\delta_i}$ and $\boldsymbol{w_i}$ is straightforward, as long as there are sufficient number of linearly independent equations as elaborated in section 3. Thanks to the reparametrization, $\boldsymbol{\delta_i}$ is revealed to be the parameter that satisfies the identification equation 7 for all S and $x, y \in S$, by *N-I*. Then, the preference parameters are revealed by

$$w_i(x) = \frac{p_i(x, X) - \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(x, X)}{\delta_i}.$$

We conclude this section with our most general stability result, that ensures convergence and uniqueness of equilibrium for the general social interaction model. All other stability results in the paper are corollaries of this.

Theorem 6. *Take* $w_i \in \Delta_{++}(x), \delta_i > 0$, *and* $\boldsymbol{\delta}_i \in \Re^{N \setminus \{i\}}$ *and*

(1) $\delta_i + \sum_{j \neq i} \delta_{ij} = 1$

(2) For every x, $\delta_i w_i(x) + \sum_{j \neq i} \min\{0, \delta_{ij}\} > 0$.

Then, there is a unique $(p_1^*,\dots,p_N^*)\in\Delta_{++}(S)^N$ *for which*

$$p_i^*(x,S)=\frac{\delta_i w_i(x)+\boldsymbol{\delta}_i\cdot\boldsymbol{p}_{-i}^*(x,S)}{\sum_{y\in S}[\delta_i w_i(y)+\boldsymbol{\delta}_i\cdot\boldsymbol{p}_{-i}^*(y,S)]}$$

and for any $(p_1^1(\cdot,S),\dots,p_N^1(\cdot,S))\in\Delta_{++}(S)^N$*, the iterative map*

$$p_i^t(x,S)=\frac{\delta_i w_i(x)+\boldsymbol{\delta}_i\cdot\boldsymbol{p}_{-i}^{t-1}(x,S)}{\sum_{y\in S}[\delta_i w_i(y)+\boldsymbol{\delta}_i\cdot\boldsymbol{p}_{-i}^{t-1}(y,S)]}$$

converges to $(p_1^*,\dots,p_N^*)$*.*

Given the w and δ parameters, an explicit representation for this unique representation is possible in terms of inverses of matrices. This expression is standard, and appears in the proof of Theorem 6. This expression demonstrates, for example, that $(p_1^*,\dots,p_N^*)\in\Delta_{++}(X)^N$ is an affine function of $(w_1,\dots,w_N)$.[25]

5. Concluding Remarks

The identification of social interactions from observable behavior is an important and highly topical agenda for economists. We believe that the use of choice theoretic tools to study social interactions introduces a new perspective to this problem that has traditionally been dealt with mostly econometrics tools.

Exploiting standard choice theoretic tools, this model, and others, should prove useful for the identification of unobservable underlying interaction structures and parameters out of observable behavior. The strength of our identification strategy relies on the novel source of variation we have introduced: the variation of the choice sets. Whether the same insight can be applied to more general settings of interaction constitutes an interesting future research avenue. One potential way to generalize our model is via more flexible definitions of individual utilities a la Luce:

$$p_i(x,S)=\frac{U_i(x|S,\alpha_i,p_j)}{\sum_{y\in S}U_i(y|S,\alpha_i,p_j)}$$

where $U_i(x|S,\alpha_i,p_j)$ represents agent i's utility when she chooses alternative x from budget S. In this paper, we aimed to come up with a particular $U_i(\cdot)$ that produces

[25]The same does not hold true for $S\subset X$, $S\neq X$ in general.

(p_1, p_2) (i) that is unique for a given set of parameters, (ii) out of which the underlying parameters can be revealed uniquely with an arguably small amount of data, (iii) that is axiomatizable, hence falsifiable, (iv) that is stable when accommodated within a dynamic adjustment process, and (v) that can be produced as the outcome of well-known interaction mechanisms such as a game, parametric social interaction models or social learning, under appropriate assumptions. Our model assumes that $U_i(x|S, \alpha_i, p_j)$ is a linear combination of the intrinsic utility and the choice probability of the other. This linearity, combined with the asymmetric role played by the self vs influence over different menus grants us the unique identification. One interesting close alternative would be

$$U_i^*(x|S, \alpha_i, p_j) = w_i(x) + \frac{\alpha_i w_i(S)}{1 - \alpha_i} p_j(x, S).$$

According to this formulation, the decision maker inherently places different weights on the choice probability of others across different menus— non-linear weighting. What makes this formulation interesting is that it boils down to a convex combination of two Luce models as follows:

$$p_i(x, S) = \lambda_i \frac{w_i(x)}{\sum_{y \in S} w_i(y)} + (1 - \lambda_i) \frac{w_j(x)}{\sum_{y \in S} w_j(y)}$$

where $\lambda_i = \dfrac{1 - \alpha_i}{1 - \alpha_i \alpha_j}$. However, this model does not always lends itself to unique identification of the underlying parameters out of observable behavior. Moreover, the non-linearity prevents the model to be interpreted as the outcome of a random utility maximization with social interactions or a Logit QRE. We hope that similar results can be obtained by studying different forms of utility, which extends the insights of this paper.

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6. Appendix

Proof of Theorem 1. ($\Rightarrow$) Let (p_1, p_2) with $p_1 \neq p_2$ have a dual interaction representation with $(w_1, w_2, \alpha_1, \alpha_2)$.

First we assume that β_i is well-defined and show that Equation 6 holds for all x, y and S. Define $\beta_i \equiv \dfrac{\alpha_i}{1+\alpha_i}$. Then $\beta_i \left(\dfrac{d_j(x,S)}{p_i(x,S)} - \dfrac{d_j(y,S)}{p_i(y,S)}\right)$ is equal to

$$
\begin{aligned}
&= \frac{\alpha_i}{1+\alpha_i}\left(\frac{p_j(x,S)-p_j(x,X)}{p_i(x,S)} - \frac{p_j(y,S)-p_j(y,X)}{p_i(y,S)}\right)\\
&= \frac{w_i(x)+\alpha_i p_j(x,S) - w_i(x) - \alpha_i p_j(x,X)}{(1+\alpha_i)p_i(x,S)} - \frac{w_i(y)+\alpha_i p_j(y,S) - w_i(x) - \alpha_i p_j(y,X)}{(1+\alpha_i)p_i(y,S)}\\
&= \frac{(w_i(S)+\alpha_i)p_i(x,S) - (1+\alpha_i)p_i(x,X)}{(1+\alpha_i)p_i(x,S)} - \frac{(w_i(S)+\alpha_i)p_i(y,S) - (1+\alpha_i)p_i(y,X)}{(1+\alpha_i)p_i(y,S)}\\
&= \frac{p_i(y,X)}{p_i(y,S)} - \frac{p_i(x,X)}{p_i(x,S)}\\
&= \frac{d_i(x,S)}{p_i(x,S)} - \frac{d_i(y,S)}{p_i(y,S)}.
\end{aligned}
$$

Since this holds for all $S \neq X$ and distinct $x, y \in S$, Equation 6 holds for all x, y and S.

Now we show that β_i is indeed well-defined. We have three exhaustive cases. Fix $i, j \in \{1, 2\}$ with $i \neq j$ and first let $\alpha_i \neq 0$. We will show that for some $S \neq X$ and distinct x, y, we have $\frac{d_j(x,S)}{p_i(x,S)} - \frac{d_j(y,S)}{p_i(y,S)} \neq 0$, hence, $\beta_i(x,y,S)$ exists. Assume for a contradiction that $\frac{d_j(x,S)}{p_i(x,S)} - \frac{d_j(y,S)}{p_i(y,S)} = 0$ for all S and distinct x, y. Then,

$$
\begin{aligned}
\frac{d_j(x,S)}{p_i(x,S)} - \frac{d_j(y,S)}{p_i(y,S)} = 0 &\Rightarrow \frac{p_j(x,S) - p_j(x,X)}{p_i(x,S)} = \frac{p_j(y,S) - p_j(y,X)}{p_i(y,S)} \\
&\Rightarrow \frac{\alpha_i p_j(x,S) - \alpha_i p_j(x,X)}{p_i(x,S)} = \frac{\alpha_i p_j(y,S) - \alpha_i p_j(y,X)}{p_i(y,S)} \\
&\Rightarrow \frac{w_i(x) + \alpha_i p_j(x,S) - w_i(x) - \alpha_i p_j(x,X)}{p_i(x,S)} = \frac{w_i(y) + \alpha_i p_j(y,S) - w_i(y) - \alpha_i p_j(y,X)}{p_i(y,S)} \\
&\Rightarrow \frac{[w_i(S) + \alpha_i] p_i(x,S) - [1 + \alpha_i] p_i(x,X)}{p_i(x,S)} = \frac{[w_i(S) + \alpha_i] p_i(y,S) - [1 + \alpha_i] p_i(y,X)}{p_i(y,S)} \\
&\Rightarrow \frac{p_i(x,X)}{p_i(x,S)} = \frac{p_i(y,X)}{p_i(y,S)}
\end{aligned}
$$

But since this holds for all S, x, y, then IIA would be satisfied, establishing a contradiction with $\alpha_i \neq 0$. Now consider $\alpha_i = 0$ and $\alpha_j \neq 0$. Then p_i has a Luce representation and $\frac{d_i(x,S)}{p_i(x,S)} - \frac{d_i(y,S)}{p_i(y,S)} = 0$ for all S and $x, y \in S$. We now show that for some S and distinct $x, y \in S$, $\frac{d_j(x,S)}{p_i(x,S)} - \frac{d_j(y,S)}{p_i(y,S)} \neq 0$ so that I is satisfied for $\beta_i = \frac{\alpha_i}{1+\alpha_i} = 0$. Assume for a contradiction that for all S and distinct $x, y \in S$, $\frac{d_j(x,S)}{p_i(x,S)} - \frac{d_j(y,S)}{p_i(y,S)} = 0$. Since $\alpha_j \neq 0$, we have

$$
\frac{\alpha_j}{1+\alpha_j} \left(\frac{d_i(x,S)}{p_j(x,S)} - \frac{d_i(y,S)}{p_j(y,S)} \right) = \frac{d_j(x,S)}{p_j(x,S)} - \frac{d_j(y,S)}{p_j(y,S)} \tag{9}
$$

for all S and distinct $x, y \in S$, as we have shown above. Take S and $x, y \in S$ with $\frac{d_i(x,S)}{p_j(x,S)} \neq \frac{d_i(y,S)}{p_j(y,S)}$ and substitute $d_j(x,S)$ by $d_j(y,S)p_i(x,S)/p_i(y,S)$ in (9):

$$
\begin{aligned}
\frac{\alpha_j}{1+\alpha_j} \left(\frac{d_i(x,S)}{p_j(x,S)} - \frac{d_i(y,S)}{p_j(y,S)} \right) &= \frac{d_j(y,S)p_i(x,S)}{p_j(x,S)p_i(y,S)} - \frac{d_j(y,S)}{p_j(y,S)} \\
\frac{\alpha_j}{1+\alpha_j} \left(\frac{d_i(x,S)p_j(y,S) - d_i(y,S)p_j(x,S)}{p_j(x,S)p_j(y,S)} \right) &= \frac{d_j(y,S)p_i(x,S)p_j(y,S) - d_j(y,S)p_j(x,S)p_i(y,S)}{p_j(x,S)p_i(y,S)p_j(y,S)} \\
\frac{\alpha_j}{1+\alpha_j} &= \frac{d_j(y,S)[p_i(x,S)p_j(y,S) - p_i(y,S)p_j(x,S)]}{p_i(y,S)[d_i(x,S)p_j(y,S) - d_i(y,S)p_j(x,S)]}
\end{aligned}
$$

As p_i has a Luce representation, $d_i(x,S) = p_i(x,S)(1-w_i(S))$. We can then simplify the expression as follows:

$$\frac{\alpha_j}{1+\alpha_j} = \frac{d_j(y,S)}{p_i(y,S)(1-w_i(S))} = \frac{d_j(y,S)}{d_i(y,S)}.$$

But then,

$$\begin{aligned}
\frac{\alpha_j}{1+\alpha_j} = \frac{d_j(y,S)}{d_i(y,S)} &\Rightarrow \frac{\alpha_j p_i(y,S) - \alpha_j p_i(y,X)}{1+\alpha_j} = p_j(y,S) - p_j(y,X) \\
&\Rightarrow \frac{w_j(y) + \alpha_j p_i(y,S) - w_j(y) - \alpha_j p_i(y,X)}{1+\alpha_j} = p_j(y,S) - p_j(y,X) \\
&\Rightarrow \frac{p_j(y,S)[w_j(S)+\alpha_j] - p_j(y,X)[1+\alpha_j]}{1+\alpha_j} = p_j(y,S) - p_j(y,X) \\
&\Rightarrow \frac{p_j(y,S)[w_j(S)+\alpha_j]}{1+\alpha_j} - p_j(y,X) = p_j(y,S) - p_j(y,X)
\end{aligned}$$

Contradiction since $w_j(S) \neq 1$.

Finally, let $\alpha_i = \alpha_j = 0$. We claim that there exists S and distinct $x, y \in S$ such that $\frac{d_j(x,S)}{p_i(x,S)} - \frac{d_j(y,S)}{p_i(y,S)} \neq 0$ so that $\beta_i = \frac{\alpha_i}{1+\alpha_i} = 0$ solves (6) for all S and distinct $x, y \in S$. Assume for a contradiction not. Since p_j allows a Luce representation, $d_j(x,S) = (1-w_j(S))p_j(x,S)$. But then, $\frac{d_j(x,S)}{p_i(x,S)} = \frac{d_j(y,S)}{p_i(y,S)}$ implies $\frac{p_j(x,S)}{p_i(x,S)} = \frac{p_j(y,S)}{p_i(y,S)}$. Since this would be the case for all S and $x, y \in S$, we would have $p_i = p_j$, contradiction. Thus, we have established I for all cases with $\beta_i \equiv \beta_i(x,y,S) = \frac{\alpha_i}{1+\alpha_i}$.

Nn follows directly. *UB* follows from $w_i(x) > 0$ for all x since $w_i(x) = (1+\alpha_i)p_i(x,X) - \alpha_i p_j(x,X)$. Then we have $\frac{p_i(x,X)}{p_j(x,X)} > \beta_i$, establishing necessity.

($\Leftarrow$) Let $p_1 \neq p_2$ satisfy the axioms. Now define $\beta_i \equiv \beta_i(x,y,S)$ by *I*. *UB* implies $\beta_i \neq 1$ since otherwise $1 < \frac{p_i(x,X)}{p_j(x,X)}$ for all $x \in X$ yields $p_i(x,X) > p_j(x,X)$, from which it follows that $1 = \sum_{x \in X} p_i(x,X) > \sum_{x \in X} p_j(x,X) = 1$, a contradiction. Since $\beta_i \neq 1$, define $\alpha_i := \frac{\beta_i}{1-\beta_i}$.

We claim that $\alpha_i \geq 0$. Observe that by *UB*, $\beta_i < 1$. Joint with *Nn*, this means $\beta_i \in [0,1)$. Hence it follows that $\alpha_i = \frac{\beta_i}{1-\beta_i} \geq 0$.

Next, we define weights for each alternative:

$$w_i(x) \equiv p_i(x,X) + \alpha_i(p_i(x,X) - p_j(x,X)).$$

Observe that $\sum_{x \in X} w_i(x) = 1$.

Now take any $S \neq X$ with distinct $x, y \in S$. Then:

$$\begin{aligned}
\frac{d_i(x,S)}{p_i(x,S)} - \frac{d_i(y,S)}{p_i(y,S)} &= \frac{\alpha_i}{1+\alpha_i}\left[\frac{d_j(x,S)}{p_i(x,S)} - \frac{d_j(y,S)}{p_i(y,S)}\right] \\
\frac{p_i(y,X)}{p_i(y,S)} - \frac{p_i(x,X)}{p_i(x,S)} &= \alpha_i\left[\frac{d_j(x,S) - d_i(x,S)}{p_i(x,S)} - \frac{d_j(y,S) - d_i(y,S)}{p_i(y,S)}\right] \\
\frac{p_i(x,X) + \alpha_i d_j(x,S) - \alpha_i d_i(x,S)}{p_i(x,S)} &= \frac{p_i(y,X) + \alpha_i d_j(y,S) - \alpha_i d_i(y,S)}{p_i(y,S)} \\
\frac{p_i(x,X) + \alpha_i d_j(x,S) - \alpha_i d_i(x,S) + \alpha_i p_i(x,S)}{p_i(x,S)} &= \frac{p_i(y,X) + \alpha_i d_j(y,S) - \alpha_i d_i(y,S) + \alpha_i p_i(y,S)}{p_i(y,S)}.
\end{aligned}$$

The last equality is obtained by adding α_i to both sides of the previous equality. Notice that as $-\alpha_i d_i(x,S) + \alpha_i p_i(x,S) = \alpha_i p_i(x,X)$, the numerators of both of the sides are nonzero. Hence:

$$\begin{aligned}
\frac{p_i(x,S)}{p_i(y,S)} &= \frac{p_i(x,X) + \alpha_i d_j(x,S) - \alpha_i d_i(x,S) + \alpha_i p_i(x,S)}{p_j(y,X) + \alpha_i d_j(y,S) - \alpha_i d_i(y,S) + \alpha_i p_i(y,S)} \\
&= \frac{p_i(x,X) + \alpha_i(p_i(x,X) - p_j(x,X)) + \alpha_i p_j(x,S)}{p_i(x,X) + \alpha_i(p_i(x,X) - p_j(x,X)) + \alpha_i p_j(x,S)} \\
&= \frac{w_i(x) + \alpha_i p_j(x,S)}{w_i(y) + \alpha_i p_j(y,S)}.
\end{aligned}$$

Observe in particular that this equality holds even in the case $x = y$. Now, for any $x, y \in S$, we have

$$p_i(y,S) = p_i(x,S)\frac{w_i(y) + \alpha_i p_j(y,S)}{w_i(x) + \alpha_i p_j(x,S)}$$

so that

$$\sum_{y \in S} p_i(y,S) = \sum_{y \in S} p_i(x,S)\frac{w_i(y) + \alpha_i p_j(y,S)}{w_i(x) + \alpha_i p_j(x,S)}.$$

Conclude

$$1 = p_i(x,S)\frac{\sum_{y \in S}(w_i(y) + \alpha_i p_j(y,S))}{w_i(x) + \alpha_i p_j(x,S)}.$$

Consequently,

$$p_i(x,S) = \frac{w_i(x) + \alpha_i p_j(x,S)}{\sum_{y\in S}(w_i(y) + \alpha_i p_j(y,S))}.$$

We finally show that $w_i(x) > 0$ for all $x \in X$. For all $x \in X$, $\dfrac{p_i(x,X)}{p_j(x,X)} > \beta_i = \dfrac{\alpha_i}{1+\alpha_i}$. Here, we obtain $(\alpha_i + 1)p_i(x,X) > \alpha_i p_j(x,X)$ for all x. Consequently, $w_i(x) = p_i(x,X) + \alpha_i[p_i(x,X) - p_j(x,X)] > 0$ for all x.

■

Derivation of stochastic choice function from utility maximization with log-logistic errors:

Let $U_i(x) = V_i(x)\varepsilon_i(x)$ where $V_i(x) = w_i(x) + \alpha_i p_j(x)$. Under the assumption that the disturbances are i.i.d. with a Log-logistic distribution (i.e., $\eta_i = \log \varepsilon_i$ follows a Type 1 extreme value distribution) with $g(\eta_i) = e^{-\eta_i} e^{-e^{-\eta_i}}$, maximization of log-utility yields:

$$\begin{aligned}
\log U_i(x) &= \log V_i(x) + \eta_i(x) \\
p_i(x) &= Prob\,(\log V_i(x) + \eta_i(x) > \log V_i(y) + \eta_i(y)),\ \ \forall y \neq x) \\
&= Prob\left(\eta_i(y) < \log\left(\frac{V_i(x)\varepsilon_i(x)}{V_i(y)}\right),\ \ \forall y \neq x\right)
\end{aligned}$$

Then for a given $\eta_i(x)$, using the cdf $G(\eta_i)$:

$$Prob\,(x|\eta_i(x)) = \prod_{y\neq x} \exp\left\{ - e^{-\log\left(\frac{V_i(x)\varepsilon_i(x)}{V_i(y)}\right)}\right\}$$

which leads to:

$$p_i(x) = \int_{-\infty}^{+\infty} \left(\prod_{y\neq x} \exp\left\{ - e^{-\log\left(\frac{V_i(x)\varepsilon_i(x)}{V_i(y)}\right)}\right\}\right) e^{-\eta_i} \exp\{-e^{-\eta_i}\} d\eta_i$$

$$p_i(x) = \int_{-\infty}^{+\infty} \left(\prod_{y} \exp\left\{ - e^{-\log\left(\frac{V_i(x)\varepsilon_i(x)}{V_i(y)}\right)}\right\}\right) e^{-\eta_i} d\eta_i$$

The second line above is observed by collecting terms in the exponent of e given that $\dfrac{V_i(x)}{V_i(x)} = 1$.

$$\begin{aligned} p_i(x) &= \int_{-\infty}^{+\infty} \exp\Big\{ -\sum_y e^{-\log\left(\frac{V_i(x)\varepsilon_i(x)}{V_i(y)}\right)} \Big\} e^{-\eta_i} d\eta_i \\ &= \int_{-\infty}^{+\infty} \exp\Big\{ -e^{-\eta_i} \sum_y e^{-\log\left(\frac{V_i(x)}{V_i(y)}\right)} \Big\} e^{-\eta_i} d\eta_i \end{aligned}$$

Apply a transformation of variables as $t = e^{-\eta_i(x)}$ such that $dt = -e^{-\eta_i(x)} d\eta_i$. Note that as η_i approaches infinity, t approaches zero, and as η_i approaches negative infinity, t becomes infinitely large.

$$\begin{aligned} p_i(x) &= \int_{\infty}^{0} -\exp\Big\{ -t\sum_y e^{-\log\left(\frac{V_i(x)}{V_i(y)}\right)} \Big\} dt \\ &= \int_{\infty}^{0} -\exp\Big\{ -t\sum_y \frac{V_i(y)}{V_i(x)} \Big\} dt \\ &= \frac{e^{-t\frac{\sum V_i(y)}{V_i(x)}}}{\frac{\sum V_i(y)}{V_i(x)}} \Bigg|_{\infty}^{0} = \frac{V_i(x)}{\sum\limits_y V_i(y)} = \frac{w_i(x) + \alpha_i p_j(x)}{\sum\limits_y (w_i(y) + \alpha_i p_j(y))} \end{aligned}$$

■

Proof of Theorem 3. ($\Rightarrow$) Let $(p_1, p_2, ..., p_n)$ be a social interaction model. For any i, define $\boldsymbol{\beta}_i \in R^{n-1}$ such that $\beta_{ij} = \dfrac{\alpha_{ij}}{1 + \sum\limits_{j\neq i} \alpha_{ij}}$ for all $j \neq i$. We will first show $\boldsymbol{\beta}_i \in \boldsymbol{\mathcal{B}}_i$.

First let $\alpha_{ij} = 0$ for all i and j with $i \neq j$. Then, for all i, p_i has a Luce representation and hence $d_i(x, S) = (1 - w_i(S))p_i(x, S)$. Moreover $\dfrac{d_i(x,S)}{p_i(x,S)} - \dfrac{d_i(y,S)}{p_i(y,S)} = 0$ for all S and distinct x, y. Hence $\boldsymbol{\beta}_i = \mathbf{0}$ is an element in $\boldsymbol{\mathcal{B}}_i$.

Now let $\boldsymbol{\alpha}_i \neq \mathbf{0}$ for some i. Take any S and any distinct $x, y \in S$. Then $\boldsymbol{\beta}_i \cdot \left(\dfrac{\boldsymbol{d}_{-i}(x,S)}{p_i(x,S)} - \dfrac{\boldsymbol{d}_{-i}(y,S)}{p_i(y,S)} \right)$ is equal to

$$=\sum_j \frac{\beta_{ij}(p_j(x,S)-p_j(x,X))}{p_i(x,S)} - \sum_j \frac{\beta_{ij}(p_j(y,S)-p_j(y,X))}{p_i(y,S)}$$

$$=\sum_j \frac{\alpha_{ij}(p_j(x,S)-p_j(x,X))}{(1+\sum_j \alpha_{ij})p_i(x,S)} - \sum_j \frac{\alpha_{ij}(p_j(y,S)-p_j(y,X))}{(1+\sum_j \alpha_{ij})p_i(y,S)}$$

$$=\frac{w_i(x)+\sum_j \alpha_{ij}p_j(x,S)-w_i(x)-\sum_j \alpha_{ij}p_j(x,X)}{(1+\sum_j \alpha_{ij})p_i(x,S)} - \frac{w_i(y)+\sum_j \alpha_{ij}p_j(y,S)-w_i(y)-\sum_j \alpha_{ij}p_j(y,X)}{(1+\sum_j \alpha_{ij})p_i(y,S)}$$

$$=\frac{p_i(x,S)[w_i(S)+\sum_j \alpha_{ij}]-p_i(x,X)[1+\sum_j \alpha_{ij}]}{(1+\sum_j \alpha_{ij})p_i(x,S)} - \frac{p_i(y,S)[w_i(S)+\sum_j \alpha_{ij}]-p_i(y,X)[1+\sum_j \alpha_{ij}]}{(1+\sum_j \alpha_{ij})p_i(y,S)}$$

$$=\frac{[w_i(S)+\sum_j \alpha_{ij}]}{(1+\sum_j \alpha_{ij})} - \frac{p_i(x,X)}{p_i(x,S)} - \frac{[w_i(S)+\sum_j \alpha_{ij}]}{(1+\sum_j \alpha_{ij})} + \frac{p_i(y,X)}{p_i(y,S)}$$

$$=\frac{p_i(y,X)}{p_i(y,S)} - \frac{p_i(x,X)}{p_i(x,S)},$$

establishing $\boldsymbol{\beta}_i \in \mathcal{B}_i$.

Certainly, $\boldsymbol{\beta}_i \in R_+^{n-1}$ as $\alpha_{ij} \geq 0$ for all i, j with $i \neq j$. *N-PUB* then follows from $w_i(x) > 0$ for all x, since $w_i(x) = p_i(x,X) + \sum_{j\neq i} \alpha_{ij}(p_i(x,X) - p_j(x,X)) > 0 \Rightarrow (1+\sum_{j\neq i}\alpha_{ij})p_i(x,X) > \sum_{j\neq i}\alpha_{ij}p_j(x,X) \Rightarrow p_i(x,X) > \boldsymbol{\beta}_i \cdot \boldsymbol{p}_{-i}(x,X)$.

($\Leftarrow$) Take $(p_1, p_2, ..., p_n)$ satisfying our axioms. Take any $i \in N$, x, y and S and by *N-I*, take $\boldsymbol{\beta}_i \in \mathcal{B}_i$, which also satisfies *N-PUB*. Further, define $\boldsymbol{\alpha}_i \in R^{n-1}$ such that $\alpha_{ij} = \dfrac{\beta_{ij}}{1-\sum_{j\neq i}\beta_{ij}}$. We first show that $\boldsymbol{\alpha}_i$ is well-defined and nonnegative since $\sum_{j\neq i}\beta_{ij} < 1$. This is because by *N-PUB*, $p_i(x,X) > \boldsymbol{\beta}_i \cdot \boldsymbol{p}_{-i}(x,X)$ for all x, we have $1 = \sum_{x\in X} p_i(x,X) > \sum_{x\in X} \boldsymbol{\beta}_i \cdot \boldsymbol{p}_{-i}(x,X) = \sum_{j\neq i}\beta_{ij}$. Hence, $\boldsymbol{\alpha}_i \in R_+^{n-1}$ is well-defined for all $\boldsymbol{\beta}_i$ as claimed.

Notice we then have $\dfrac{1}{1+\sum_{j\neq i}\alpha_{ij}}\boldsymbol{\alpha}_i = \boldsymbol{\beta}_i$. Now define

$$w_i(x) := p_i(x,X) + \boldsymbol{\alpha}_i \cdot [p_i(x,X)\mathbf{1} - \boldsymbol{p}_{-i}(x,X)]$$

where $\mathbf{1} \in R^{n-1}$ is a vector of ones and observe that

$$\begin{aligned}
\sum_{x\in X} w_i(x) &= \sum_{x\in X} \left(p_i(x,X) + \boldsymbol{\alpha}_i \cdot [p_i(x,X)\mathbf{1} - \boldsymbol{p}_{-i}(x,X)]\right) \\
&= 1 + \boldsymbol{\alpha}_i \cdot \left[\sum_{x\in X} p_i(x,X)\mathbf{1} - \sum_{x\in X} \boldsymbol{p}_{-i}(x,X)\right] \\
&= 1 + \boldsymbol{\alpha}_i(\mathbf{1} - \mathbf{1}) \\
&= 1.
\end{aligned}$$

By *N-I*,

$$\frac{1}{1+\sum_{j\neq i}\alpha_{ij}}\boldsymbol{\alpha}_i \cdot \left(\frac{\boldsymbol{d}_{-i}(x,S)}{p_i(x,S)} - \frac{\boldsymbol{d}_{-i}(y,S)}{p_i(y,S)}\right) = \frac{p_i(y,X)}{p_i(y,S)} - \frac{p_i(x,X)}{p_i(x,S)}$$

$$\begin{aligned}
\frac{(1+\sum_{j\neq i}\alpha_{ij})p_i(x,X) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(x,S) - \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(x,X)}{p_i(x,S)} &= \frac{(1+\sum_{j\neq i}\alpha_{ij})p_i(y,X) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(y,S)}{p_i(y,S)} \\
&\quad - \frac{\boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(y,X)}{p_i(y,S)}.
\end{aligned}$$

Notice that numerators in both of the sides are positive since $p_j(x,S) > p_j(x,X)$ for all j, x and S. Hence

$$\begin{aligned}
\frac{p_i(x,S)}{p_i(y,S)} &= \frac{p_i(x,X) + \boldsymbol{\alpha}_i \cdot [p_i(x,X)\mathbf{1} - \boldsymbol{p}_{-i}(x,X)] + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(x,S)}{p_i(y,X) + \boldsymbol{\alpha}_i \cdot [p_i(y,X)\mathbf{1} - \boldsymbol{p}_{-i}(y,X)] + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(y,S)} \\
&= \frac{w_i(x) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(x,S)}{w_i(y) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(y,S)}.
\end{aligned}$$

But then, since this claim holds for all $y \in S$:

$$\begin{aligned} p_i(y,S) &= p_i(x,S)\frac{w_i(y) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(y,S)}{w_i(x) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(x,S)} \\ \sum_{y\in S} p_i(y,S) &= \sum_{y\in S} p_i(x,S)\frac{w_i(y) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(y,S)}{w_i(x) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(x,S)} \\ 1 &= p_i(x,S)\frac{\sum_{y\in S}[(w_i(y) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(y,S)]}{w_i(x) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(x,S)} \\ p_i(x,S) &= \frac{w_i(x) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(x,S)}{\sum_{y\in S}[w_i(y) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(y,S)]}. \end{aligned}$$

We finally show that $w_i(x) > 0$ for all $x \in X$. This is established by *N-PUB*. Since $p_i(x,X) > \boldsymbol{\beta}_i \boldsymbol{p}_{-i}(x,X)$ and $1+\sum_{j\neq i} \alpha_{ij} > 0$, then, $(1+\sum_{j\neq i} \alpha_{ij})p_i(x,X) > \boldsymbol{\alpha}_i \boldsymbol{p}_{-i}(x,X) \Rightarrow w_i(x) > 0$. ■

General Social Interaction Model: Derivation of parametric bounds

With negative interactions in play, existence and stability are not straightforward implications of the model. In particular certain parametric restrictions are required to ensure that the linear aggregation procedure defines a probability, and the model remains meaningful for the dynamic adjustment procedure of subsection 2.4.

First, let $n = 2$ and consider the two indiviudal model introduced in section 4:

$$p_i(x,S) = \frac{\delta_i w_i(x) + (1-\delta_i)p_j(x,S)}{\delta_i w_i(S) + 1 - \delta_i}.$$

In order to find the bounds on the parameters, first suppose that $p_j(x,S) = 0$. It follows that

$$p_i(x,S) = \frac{\delta_i w_i(x)}{\delta_i w_i(S) + 1 - \delta_i}.$$

This term will be non-negative and well-defined exactly when $\delta_i w_i(S) + 1 - \delta_i > 0$. So, this is a first necessary condition for all S. As a second observation, suppose that $p_j(x,S) = 1$. It follows that $p_i(x,S) = \frac{\delta_i w_i(x) + 1 - \delta_i}{\delta_i w_i(S) + 1 - \delta_i}$. Given that $\delta_i w_i(S) + 1 - \delta_i > 0$, this term will be nonnegative exactly when $\delta_i w_i(x) + 1 - \delta_i \geq 0$. Observe that $\delta_i w_i(x) + 1 - \delta_i \geq 0$ for each x already implies the first condition, hence making it redundant. Thus with two agents, $\delta_i > 0$ is required to have a meaningful interpretation of negative influence and the condition that $\delta_i w_i(x) + 1 - \delta_i \geq 0$ for each x ensures that p_i remains a well-defined probability.

Now let $n > 2$ and consider the general social interactions model. Similar to the two agent case, $\delta_i > 0$ is necessary to ensure a meaningful representation with negative influence. This is the first requirement for the validity of the general model. In order to ensure that, no matter what $\boldsymbol{p}_{-i}(x, S)$ is, $p_i(x, S)$ is a well-defined probability, the condition we require is

$$\delta_i w_i(x) + \sum_{j \neq i} \min\{0, \delta_{ij}\} \geq 0$$

for all x, and in particular, for every S, that there exists some x for which the inequality is strict. To see the reasoning under this condition, first imagine that $\boldsymbol{p}_{-i}(x, S) = \mathbf{0}$. Then $p_i(x, S) = \dfrac{\delta_i w_i(x)}{\delta_i w_i(S) + \sum_{j \neq i} \delta_{ij}}$. Consequently, for $\delta_i > 0$, it follows that $\delta_i w_i(S) + \sum_{j \neq i} \delta_{ij} > 0$ ensures a positive $p_i(x, S)$. Second, suppose that $\boldsymbol{p}_{-i}(x, S) = \mathbf{1}_{\{j : \delta_{ij} < 0\}}$. This then implies that $p_i(x, S) = \dfrac{\delta_i w_i(x) + \sum_{j \neq i} \min\{0, \delta_{ij}\}}{\delta_i w_i(S) + \sum_{j \neq i} \delta_{ij}}$. Hence, jointly with the previous condition, to have a well-defined probability, we must ensure that

$$\delta_i w_i(x) + \sum_{j \neq i} \min\{0, \delta_{ij}\} \geq 0$$

for all x, and in particular, for every S, that there exists some x for which the inequality is strict. In particular, it is only a slight loss of generality to assume that the inequality is strict *for every* x. Thus, for all x, we have:

$$\delta_i w_i(x) + \sum_{j \neq i} \min\{0, \delta_{ij}\} > 0. \tag{10}$$

Equation (10) ensures that every profile of probability measures of the other agents is mapped to one with full support. Hence, it is a sufficient and almost necessary condition for the dynamic adjustment procedure to always result in a probability measure, providing existence. It is necessary and sufficient to always map any probability measure into a full-support probability measure. As we establish in Theorem 6, it also provides the convergence of the dynamic adjustment process.

Equation (10) has a very simple interpretation. Recall that the numerator of the expression defining choice reflects the relative propensity to choose. Equation (10) requires that this propensity to choose be positive, independently of the choices of others. ■

Proof of Theorem 5. ($\Rightarrow$) Let $(p_1, p_2, ..., p_n)$ be a general social interaction model. We will first show $\boldsymbol{\delta}_i \in \mathcal{B}_i$.

First let $\delta_{ij} = 0$ for all i and j with $i \neq j$. Then, for all i, p_i has a Luce representation and hence $d_i(x,S) = (1 - w_i(S))p_i(x,S)$. Moreover $\dfrac{d_i(x,S)}{p_i(x,S)} - \dfrac{d_i(y,S)}{p_i(y,S)} = 0$ for all S and distinct x, y. Hence $\boldsymbol{\delta}_i = \mathbf{0}$ is an element in $\mathcal{B}_i$.

Now let $\boldsymbol{\delta}_i \neq \mathbf{0}$ for some i. Take any S and any distinct $x, y \in S$. Then $\boldsymbol{\delta}_i \cdot \left(\dfrac{\boldsymbol{d}_{-i}(x,S)}{p_i(x,S)} - \dfrac{\boldsymbol{d}_{-i}(y,S)}{p_i(y,S)}\right)$ is equal to

$$
\begin{aligned}
&= \sum_j \frac{\delta_{ij}(p_j(x,S) - p_j(x,X))}{p_i(x,S)} - \sum_j \frac{\delta_{ij}(p_j(y,S) - p_j(y,X))}{p_i(y,S)} \\
&= \frac{\delta_i w_i(x) + \boldsymbol{\delta}_i \boldsymbol{p}_{-\boldsymbol{i}}(x,S) - \delta_i w_i(x) - \boldsymbol{\delta}_i \boldsymbol{p}_{-\boldsymbol{i}}(x,X)}{p_i(x,S)} - \frac{\delta_i w_i(y) + \boldsymbol{\delta}_i \boldsymbol{p}_{-\boldsymbol{i}}(y,S) - \delta_i w_i(y) - \boldsymbol{\delta}_i \boldsymbol{p}_{-\boldsymbol{i}}(y,X)}{p_i(y,S)} \\
&= \frac{p_i(x,S)[\delta_i w_i(S) + \sum_j \delta_{ij}] - p_i(x,X)}{p_i(x,S)} - \frac{p_i(y,S)[\delta_i w_i(S) + \sum_j \delta_{ij}] - p_i(y,X)}{p_i(y,S)} \\
&= \frac{p_i(y,X)}{p_i(y,S)} - \frac{p_i(x,X)}{p_i(x,S)},
\end{aligned}
$$

establishing $\boldsymbol{\delta}_i \in \mathcal{B}_i$.

Since $\delta_i > 0$, we have $\displaystyle\sum_j \delta_{ij} < 1$. *GN-UB* then follows from $\delta_i w_i(x) + \displaystyle\sum_{j \neq i} \min\{0, \delta_{ij}\} >$ 0 for all x, since $\delta_i w_i(x) = p_i(x,X) - \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(x,S)$.

($\Leftarrow$) Take $(p_1, p_2, ..., p_n)$ satisfying our axioms. Take any $i \in N$, x, y and S and by *N-I*, take $\boldsymbol{\delta}_i \in \mathcal{B}_i$, which also satisfies *GN-UB*. Let $\delta_i = 1 - \sum \delta_{ij}$. Notice $\delta_i > 0$ by *GN-UB*.

Now define

$$w_i(x) := \frac{p_i(x,X) - \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(x,X)}{\delta_i}$$

and observe that

$$\sum_{x\in X} w_i(x) = \frac{1-\sum \delta_{ij}}{\delta_i} = 1.$$

GN-UB then ensures that $\delta_i w_i(x) + \sum_{j\neq i} \min\{0, \delta_{ij}\} > 0$ and hence $w_i(x) > 0$ for all $x \in X$.

By *N-I*,

$$\boldsymbol{\delta}_i \cdot \left(\frac{\boldsymbol{d}_{-i}(x,S)}{p_i(x,S)} - \frac{\boldsymbol{d}_{-i}(y,S)}{p_i(y,S)}\right) = \frac{p_i(y,X)}{p_i(y,S)} - \frac{p_i(x,X)}{p_i(x,S)}$$

$$\frac{p_i(x,X) + \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(x,S) - \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(x,X)}{p_i(x,S)} = \frac{p_i(y,X) + \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(y,S) - \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(y,X)}{p_i(y,S)}.$$

Notice that numerators in both of the sides are positive by *GN-UB*. Hence

$$\begin{aligned}\frac{p_i(x,S)}{p_i(y,S)} =& \frac{p_i(x,X) + \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(x,S) - \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(x,X)}{p_i(y,X) + \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(y,S) - \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(y,X)} \\ =& \frac{\delta_i w_i(x) + \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(x,S)}{\delta_i w_i(y) + \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(y,S)}.\end{aligned}$$

But then, since this claim holds for all $y \in S$, as before, we arrive at

$$p_i(x,S) = \frac{\delta_i w_i(x) + \boldsymbol{\delta_i} \cdot \boldsymbol{p}_{-i}(x,S)}{\sum_{y\in S}[\delta_i w_i(y) + \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(y,S)]}$$

establishing the proof. ■

Proof of Theorems 2, 4, and 6. Let us suppose without loss that $|S| \geq 2$.

We use the standard parametrization with $\alpha_i^j \equiv \dfrac{\delta_{ij}}{\delta_i}$, in the case of Theorem 6.

Given any affine function $f(x) = Ax + b$, where $x \in \mathbb{R}^{|N||S|}$, $A \in \mathbb{R}^{|N||S|\times|N||S|}$, and $b \in \mathbb{R}^{|N||S|}$. It is well-known that there is a unique $x^* \in \mathbb{R}^{|N||S|}$ for which for any x^1, the process $x^t = f(x^{t-1})$ converges to x^* if the maximal absolute value of an eigenvalue of A has value less than 1. See for example, Varga (1962), Theorem 1.4. We will show that, in our case, this unique fixed point will be a member of $\Delta(S)^N$, because $f(\Delta(S)^N) \subseteq \Delta(S)^N$.

To this end, let us describe the matrix A and vector b in which we take interest. To ease the exposition, let $\hat{\alpha}_{ij} = \dfrac{\alpha_{ij}}{w_i(S) + \sum_{j \neq i} \alpha_{ij}}$ for $j \neq i$, and $\hat{w}_i(x) = \dfrac{w_i(x)}{w_i(S) + \sum_{j \neq i} \alpha_{ij}}$. Here, each 0 is the $S \times S$ matrix of zeroes, and I denotes the identity matrix in $S \times S$.

Now, we let the matrix $A = \begin{bmatrix} 0 & \hat{\alpha}_{12}I & \dots & \hat{\alpha}_{1n}I \\ \hat{\alpha}_{21}I & 0 & \dots & \hat{\alpha}_{2n}I \\ \vdots & \vdots & & \vdots \\ \hat{\alpha}_{n1}I & \hat{\alpha}_{n2}I & \dots & 0 \end{bmatrix}$ and let $b = \begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \vdots \\ \hat{w}_n \end{bmatrix}$, so that the iterated vector is of the form $p^t = \begin{bmatrix} p_1^t \\ p_2^t \\ \vdots \\ p_n^t \end{bmatrix}$.

Finally, by Corollary 1 on p. 17 of Varga (1962), we conclude that the maximal absolute value of an eigenvalue is bounded above by $\max_i \sum_{j \neq i} |\hat{\alpha}_{ij}|$. But for each i, we know that $\sum_{j \neq i} |\hat{\alpha}_{ij}| = \sum_{j \neq i} \dfrac{|\alpha_{ij}|}{w_i(S) + \sum_{j \neq i} \alpha_{ij}}$. Now, by assumption, and since $|S| \geq 2$, $0 < w_i(S) + 2\sum_{j \neq i} \min\{0, \alpha_{ij}\}$.[26] Observe then that, by adding to each side of this strict inequality $\sum_{j \neq i} |\alpha_{ij}|$ we obtain $\sum_{j \neq i} |\alpha_{ij}| < w_i(S) + \sum_{j \neq i} \alpha_{ij}$. Therefore, by definition of $\hat{\alpha}$, we conclude $\sum_{j \neq i} |\hat{\alpha}_{ij}| < 1$, which is what we wanted to show.

[26]Let $x, y \in S$ for which $x \neq y$. Then $w_i(x) + \sum_{j \neq i} \min\{0, \alpha_{ij}\} > 0$ and $w_i(y) + \sum_{j \neq i} \min\{0, \alpha_{ij}\} > 0$, so that $0 < w_i(x) + w_i(y) + 2\sum_{j \neq i} \min\{0, \alpha_{ij}\} \leq w_i(S) + 2\sum_{j \neq i} \min\{0, \alpha_{ij}\}$.

As a last point, we observe that the solution p^* is the unique vector satisfying $p^* = Ap^* + b$, or in other words, $p^* = (I - A)^{-1}b$. ■