Evaluating Ordinal Inequalities Between Groups

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Abstract

We explore the inequality measurement of an ordinal categorical variable between social groups. Our methodology is built on adapting well-known principles of cardinal inequality measurement such as Pigou-Dalton transfers, Lorenz dominance and the link to the Gini Index, to the ordinal inequality between groups setting. These principles lead us to the Net Difference Index (Lieberson, 1976). Net Difference Index makes use of rank-domination to evaluate the discrepancy between the distributions of two social groups over ordered categories. Specifically, it is equal to the difference between the probabilities that on a random selection of two individuals from two groups, the member of one of the groups occupies a higher rank than the counter group member. We provide a novel characterization of this index based on reasonable properties.

Keywords: Between-Group Inequality; Ordinal Inequality; Inequality Measurement.

JEL classification numbers: D63, I14, I24, J15, J16.

Inequality measurement of ordinal variables have received a major attention in the last two decades, as the importance of non-income variables in determining societal wellbeing has been widely acknowledged (Allison and Foster, 2004; Naga and Yalcin, 2008; Kobus, 2011; Lazar and Silber, 2013; Lv et al., 2015; Cowell and Flachaire, 2017; Gravel et al., 2020). We contribute to this literature by analysing inequality measurement with two critical aspects: First, we focus on between-group inequalities. Rather than evaluating the overall distribution of a variable in the society, we investigate how to quantify the discrepancies

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between distributions of a variable among social groups. Second, our variable of interest is of ordinal categorical nature. Hence, we explore the inequality measurement of an ordinal categorical variable between social groups.

Being crucial constructs for social conflict and unrest, between-group inequalities are considered to be important determinants of social and economic welfare (Langer, 2005; Ostby, 2008; Stewart, 2010). However, unlike between-group income inequality measurement (Bourguignon, 1979; Shorrocks, 1980; Cowell, 1980; Elbers et al., 2008), measurement of non-income between-group inequalities have not received a systematic treatment in the form of a progressively developing literature. Instead, in different strands of research, such as statistical sociology (Gastwirth, 1975; Lieberson, 1976; Blackburn et al., 2001), segregation measurement (Hutchens, 2006; Reardon, 2009; del Río and Alonso-Villar, 2012) or dissimilarity measurement (Andreoli and Zoli, 2014), both for empirical and theoretical purposes, stand alone tools are developed for the assessment of the uneven distribution of non-income variables between groups, such as educational attainment, health, occupational status or subjective well-being.[1] Shooting at this gap, our aim is to develop a justified framework to evaluate ordinal inequalities between groups that is based on foundational analysis. Our methodology is to draw analogies and adopt principles from well-established tools of cardinal inequality measurement such as Pigou-Dalton transfers, Lorenz ordering and its link to the Gini Index and adapt those to the particulars of our setting. Hence we demonstrate a similar approach to Le Breton et al. (2012) of discrimination measurement and to Hutchens (1991) of segregation measurement literatures.

A quick fix to the inequality measurement of ordinal variables has been to transform these ordinal variables to cardinal ones by using specific cardinalisations in order to enable the use of measures of income inequality.[2] However as first shown by Allison and Foster (2004) application of cardinal measures over these ordinal variables might result in incomparable levels of inequalities for different societies since these techniques are sensitive to scale changes,

[1]These non-income variables are not necessarily cardinal in nature; instead they define ordinal categories. Educational attainment, the highest level of education attained is a widely used indicator, standardized by the International Standard Classification of Education (ISCED) by UNESCO. It is considered to be an improvement over 'years of schooling' since it accounts for different duration of analogous school cycles in different countries (Meschi and Scervini, 2011). The data on health and subjective well-being are collected via nation-wide surveys. For practical purposes these variables are either defined over ordered categories such as 'poor, fair, good, excellent' or over a scale such as '1,2,3,4', where 1 corresponding to 'poor', '2' to 'fair' and so on.

[2]For instance, in the measurement of inequality in educational attainment, although the data is collected over attainment categories, a common practice has been to assign the average number of years of schooling to corresponding categories (Barro and Lee, 1993, 1996, 2001; Thomas et al., 2001).

i.e.; once the scale changes, measured inequality changes.[3] Certainly our setting is not immune to the same problem. Consider the following example:

Example: Society A and Society B consist of 100 women and 100 men with the below distributions over 4 categories of an ordinal variable. Let us assume that a researcher wants to adopt a well-known measure of between group income inequality, Between Group Gini Index (GG)[4] and hence makes use of a specific cardinalisation of this ordinal variable that assigns each category a value from 0 to 50.[5] The corresponding scale is given by the first columns of the tables above, where the second and third columns denote the number of individuals from each social group in each category.

Society A		
Categories	Women	Men
4^{th} Level: 47	50	40
3^{rd} Level: 32	10	20
2^{nd} Level: 21	12	20
1^{st} Level: 14	28	20

Society B		
Categories	Women	Men
4^{th} Level: 47	55	50
3^{rd} Level: 32	10	20
2^{nd} Level: 21	12	20
1^{st} Level: 14	23	10

The researcher concludes that there is more inequality between groups in Society A than B since $GG(A) = 0.007193144 > GG(B) = 0.005050505$. However if a different cardinalisation had been used, where the scaling is approximated to a single digit, i.e., 4 instead of 47, 3 instead of 32 and so on, the conclusion would have been that the between group inequality is higher in Society B since $GG(A) = 0.001779359 < GG(B) = 0.010708402$.

There exists a need for going beyond measures of income inequality and developing justified measurement methodologies for the evaluation of these non-income inequalities between social groups. That is what we aim to do. We focus on two social groups and first suggest simple tools enabling us to compare societies unambiguously in terms of the ordinal between-group inequality they possess. *Dominance weakening transfers* and the *Dominance curve* make use of stochastic dominance to compare societies and they can be seen as analogous to Pigou-Dalton transfers and Lorenz curve of the income inequality measurement.

[3]One strategy that has been developed by the literature is to come up with specific cardinalisations that are immune to scale changes so that measured inequality becomes invariant to scale. See Naga and Yalcin (2008), Kobus and Milos (2012) and Cowell and Flachaire (2017) for more on this approach.

[4]Between Group Gini Index is computed by replacing the income values (in this case, cardinal scores) of each group member by the mean income of their respective group. For this example with two groups, it can be computed as $GG = \frac{WM|\mu_W - \mu_M|}{(W+M)^2\mu}$, where W, M denote the population of Women and Men, respectively; μ values stand for the mean scores of Women, Men and the total population respectively.

[5]For instance mean age, the median number of years of schooling and experience, average life expectancy, occupational prestige score, etc...

Naturally, they do not provide a complete ranking of societies. To this purpose, mimicking the relationship of the Gini index to the Lorenz curve, integration of the Dominance curve leads us to the Net Difference Index (Lieberson, 1976). The main novel contribution of the paper is the characterization of the Net Difference Index by a set of reasonable properties for an ordinal between-group inequality measure.

Net Difference Index makes use of rank-domination to evaluate the discrepancy between the distributions of two social groups over ordered categories. Specifically, it is equal to the average difference between number of dominations by groups, where domination is defined as occupying a higher ranked position than a counter-group member.[6] Intuitively the average number of dominations by a group is equal to the probability that a randomly chosen member occupies a higher rank than a randomly chosen counter-group member. Gastwirth (1975) suggests the use of this probability as a measure of earning differentials between genders, yielding Gastwirth's Discrimination Index.[7] Essentially, Net Difference Index evaluates the ordinal inequality between two groups as the difference between their respective (discretized) Gastwirth indices.

The characterization of the Net Difference Index is provided by 4 properties: Strong Transfers that ensure a monotonic response of the index to certain transfers; Directionality that is responsible from symmetric comparison between the groups around 0; Successive Proportional Merges that accounts for invariance to the merges of adjacent positions with the same between group ratios; and finally, Decomposability that allows for overall inequality to be expressed as a weighted average of the inequalities in subparts of the society. We discuss the significance of each of these properties for the behavior of Net Difference Index in Subsection 2.2 and propose related indices that satisfy all but one of the stated properties. We pay particular attention to Directionality and devote an entire subsection, Subsection 2.3, to the characterization of a new variation of the index where Directionality is replaced with a property that ensures symmetric treatment of the groups.

To the best of our knowledge, this is the first paper to fully characterize an index of between-group inequality designed for ordinal categorical variables. The closest work from

[6]Net Difference Index is based on Mann-Whitney's U Statistics (Mann and Whitney, 1947), which gives a non-parametric rank test that is used to determine if two samples are from the same population. The Statistics U is simply the number of times the observations from one sample precede the observations from the other sample when all of the observations are ordered into a single ranked series. The probability distribution tables of U are provided for testing the null hypothesis that two samples share the same distribution. The Statistics U is different from well-known Wilcoxon rank-sum statistics (Wilcoxon, 1945) only in that U allows for different sample sizes.

[7]For a detailed analysis of how Gastwirth measure relates to stochastic dominance, see Le Breton et al. (2012).

the literature in terms of methodology and purposes (axiomatic characterization of a method to evaluate the discrepancy of group distributions over ordered categories) can be found in Andreoli and Zoli (2014). As a part of a larger research agenda that links segregation, ordinal inequality and discrimination, they propose an ordering of societies according to the discrepancy of the group distributions over ordered categories. They call this notion 'dissimilarity preserving ordinal information' and the main difference with the between-group ordinal inequality measurement principles we have in this paper comes from an axiom, Interchange of Groups, that allows to swap group distributions for certain sets of adjacent positions. This basically implies separability of the evaluation across positions, a property that is not satisfied by the Net Difference Index, simply because at each position, not only the distributions at that position matter, but the distribution of the lower ranked or higher ranked counter-group members is equivalently important. We believe this is a desirable property for an ordinal inequality measure. It is worth adding that the dissimilarity ordering also respects Successive Proportional Merges (named Independence from Split of Classes in that work).

A related strand of research that explores the uneven distribution of social groups across ordered categories comes from the literature on ordinal segregation. In a seminal paper, Reardon (2009) conceptualizes ordinal segregation as 'the extent to which variation within social groups is less than total variation in the population', and suggests several indices that depend on the distances of the distributions of groups to a completely polarized distribution. This paper does not present any characterizations, but suggests a set of properties for this setting, that are not necessarily appropriate for our question of between group inequality. This is because the main focus of segregation for that work is how the distribution within each social group compares to the distribution in the society, rather than how social groups compare to each other.

A final related line of work originates from the decomposability of ordinal inequality measures (Allison and Foster, 2004; Naga and Yalcin, 2008; Kobus and Milos, 2012; Dutta and Foster, 2013). Although these measures are developed to measure the overall inequality of an ordinal variable, they might possess decomposability properties that allow the overall inequality to be expressed as an aggregation of the inequalities within groups and between groups. Then a comparison of their between-group counterpart to our methodology would be relevant. Kobus and Milos (2012) provide a characterization of a decomposable family of indices that respect Allison and Foster partial ordering (Allison and Foster, 2004). However their decomposability property does not allow for between-group comparisons; instead it aggregates inequality values within subgroups, weighted by subgroup sizes. Thus the indices that belong to this family, the Absolute Value Index of Naga and Yalcin (2008) and Apouey (2007), do not possess between-group inequality counterpart. Dutta and Foster (2013) decompose the overall inequality of happiness (as quantified by self reported subjective well-being data) in the US over groups of race, gender and region, by using the Allison-Foster index (AF)(Allison and Foster, 2004). They, too, end up without any between-group inequality since the median category for all of the social groups happens to be the same (the data comes over 3 happiness categories, and hence it is not unreasonable that all social groups have their median reporting in the second category). When all groups have the same median, AF expresses overall inequality as a weighted sum of the inequalities *within* subgroups, just like the decomposability considered in Kobus and Milos (2012). Decomposability of AF with different subgroup medians have not been explored.

In the following section, we introduce the basic set up and the preliminaries of comparing societies with respect to the ordinal between group inequality. Section 2 introduces the Net Difference Index. We provide a set of properties and the foundational analysis of the Net Difference Index in this section. Subsection 2.2 discusses the independence of characterizing properties as well as related indices. We devote Subsection 2.3 to the version of the index without directionality property. Finally, Section 3 concludes with possible extensions of the framework. All proofs are left to an appendix.

1 The Setting

Consider an ordinal variable with finite number of categories. Let us call each category of this variable as a 'position'. Let n denote the number of positions. The ordering of the positions is exogenous and known. For positions $1, 2, 3, ..., n$, we adopt the convention that 1 is a better position than 2, which is a better position than 3 and so on. We denote a generic position by i or j so that $i < j$ implies i is a better position than j. A society $S \in C = \cup_{n \in \mathbb{Z}_{++}} \mathbb{R}_{+}^{n \times 2}$ consists of two social groups, say Women and Men, distributed over n ordered positions.[8] Let W_i and M_i denote the number of Women and Men in position i, respectively, with $(W_1, W_2, ..., W_n) = \mathbf{W}^T$, $(M_1, M_2, ..., M_n) = \mathbf{M}^T$ (T stands for transpose) and $\sum_{i=1}^{n} W_i = W$, $\sum_{i=1}^{n} M_i = M$. Assume, $W > 0$ and $M > 0$. Then $S = (\mathbf{W}, \mathbf{M})$ represents a society where the first column corresponds to the distribution of the population of women over ordered positions and the second column shows that of men. When convenient, frequencies are used,

[8] $\mathbb{R}_{+}^{n \times 2}$ refers to $n \times 2$ Real matrices with nonnegative entries. Use of Real domain is not an uncommon practice in measurement literature. For instance, part-time workers might be treated as fractional workers, etc. (Hutchens, 1991; Andreoli and Zoli, 2014).

denoted by $w_i = \frac{W_i}{W}$, $m_i = \frac{M_i}{M}$ and $(w_1, w_2, ..., w_n) = \mathbf{w}^T$, $(m_1, m_2, ..., m_n) = \mathbf{m}^T$.

1.1 A Partial Ranking: Dominance Preorder

We first aim to present an unambiguous ranking criteria, just like Pigou-Dalton transfers and Lorenz ordering of income inequality, for our setting. Given the ordinality of the variable of interest, stochastic dominance is a most natural reference, as it is scale independent. The distribution of Women first order stochastically dominates that of Men, $\mathbf{W} \succ^{SD} \mathbf{M}$, if for any position i, the proportion of women occupying positions at least at good as i are never less than that of men; i.e.; for any i, $\sum_1^i w_i \geq \sum_1^i m_i$ with at least one strict inequality.

We define a **dominance weakening transfer** as promoting members of the dominated group or demoting members of the dominant group without eliminating the stochastic dominance. Specifically, if $\mathbf{W} \succ^{SD} \mathbf{M}$, then any transfer of mass $\delta > 0$ such that $\mathbf{W}' = (W_1, ..., W_i - \delta, ..., W_j + \delta, ..., W_n)$ (or $\mathbf{M}' = (M_1, ..., M_i + \delta, ..., M_j - \delta, ..., M_n)$) for some $i < j$ with $\mathbf{W}' \succ^{SD} \mathbf{M}$ (or $\mathbf{W} \succ^{SD} \mathbf{M}'$) is a dominance weakening transfer. Similarly, if $\mathbf{M} \succ^{SD} \mathbf{W}$, then promoting women from j to i or demoting men from i to j for some $i < j$ by preserving the stochastic dominance is a dominance weakening transfer.

Clearly dominance weakening transfers suggest a very natural ordering of societies in terms of the inequality between groups as does Pigou-Dalton transfers for inequality of income between individuals. Let us introduce a graphical representation of this. Consider the following societies distributed over 3 positions as follows:

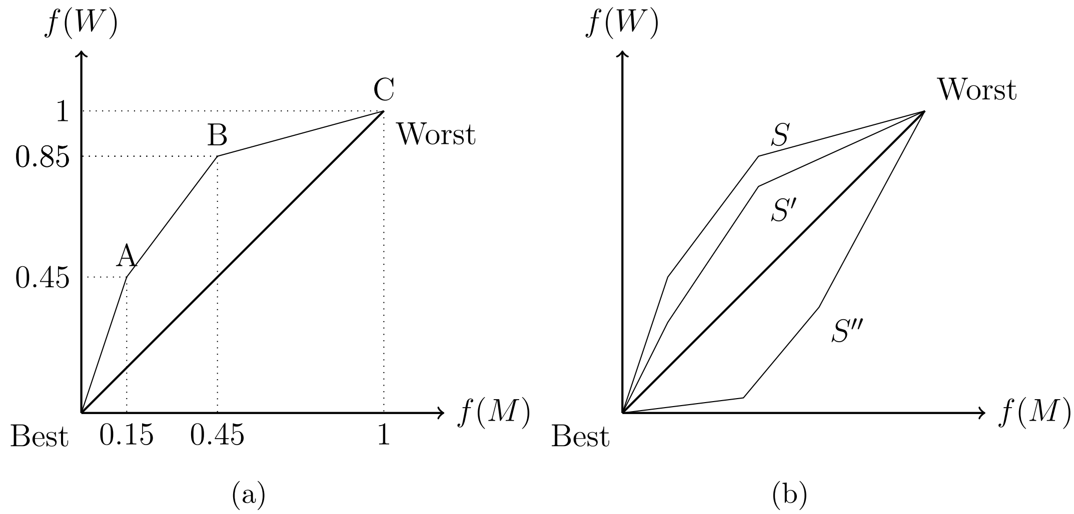
$$S = \begin{pmatrix} 45 & 15 \\ 40 & 30 \\ 15 & 55 \end{pmatrix} \quad S' = \begin{pmatrix} 30 & 15 \\ 45 & 30 \\ 25 & 55 \end{pmatrix} \quad S'' = \begin{pmatrix} 5 & 40 \\ 30 & 25 \\ 65 & 35 \end{pmatrix}$$

Figure (1a) plots the cumulative frequency distribution, $f : R_+ \to [0, 1]$ of Men against that of Women for S. Similar to the logic of the Lorenz curve, the individuals are ordered in line with their positions from best to worst. Point A corresponds to the cumulative frequency of the individuals of the first position only, whereas point B marks the cumulative frequency of the first two positions. Finally, at point C all individuals are considered.

We call this curve **the Dominance curve** as it can be interpreted as a visualisation of the stochastic dominance between groups.[9] Formally, the Dominance curve for S is given

[9]The notion of using a mapping of cumulative distributions to assess the discrepancy or similarity of two populations is nowhere novel to this paper. The Dominance curve is conceptually equivalent to the Segregation curve (Duncan and Duncan, 1955), the Concentration curve (Mahalanobis, 1960) or the Discrimination curve (Le Breton et al., 2012). We present this notion merely as another foundation to the index we characterize.

Figure 1: Dominance Curve



by: $f^S : [0,1] \to [0,1]$, $f^S(m) = \sum_{i=1}^{k-1} w_i + w_k \frac{\Delta m}{m_k}$, where $k = \max\{1,..,i,..,n\}$ such that $m = \sum_{i=1}^{k-1} m_i + \Delta_m$ with $\Delta_m \geq 0$. Basically, we assume uniform distribution of the groups within positions. Certainly, $f^S(\sum_{i=1}^{k} m_i) = \sum_{i=1}^{k} w_i$ for any $k \in \{1, 2, ..., n\}$. That is, $f^S(m)$ gives out the cumulative frequency of Women occupying positions that are at least as good as those m of Men. Figure (1b) depicts the Dominance curves for S' and S'' as well as S. The 45% line is the **equality line**; the Dominance curve of a society lies exactly on the equality line if and only if the frequency distribution of Women and Men are identical. If $\mathbf{W} \succ^{SD} \mathbf{M}$, as it is in S, then the Dominance curve lies fully above the equality line. Conversely, since in S'', $\mathbf{M}'' \succ^{SD} \mathbf{W}''$, the Dominance curve lies fully below the equality line. Moreover, the distance to the equality line bears a sense of the intensity of stochastic dominance: the further away from the line a society is, the higher the level of stochastic dominance between groups. Consider the society matrices S and S': We have $\mathbf{M}' = \mathbf{M}$ and we can reach from S to S' by a series of dominance weakening transfers among Women. That is exactly why S' is closer to the equality line than S.

Let us formalize this notion of being closer to the equality line with an ordering relation. Define **the Dominance preorder**, $\succ^D \in (C \times C)$ such that for S, S' with $\mathbf{W} \succ^{SD} \mathbf{M}$ and $\mathbf{W}' \succ^{SD} \mathbf{M}'$ (or $\mathbf{M} \succ^{SD} \mathbf{W}$ and $\mathbf{M}' \succ^{SD} \mathbf{W}'$), $f^S(m) \geq f^{S'}(m)$ ($f^S(m) \leq f^{S'}(m)$, respectively) for all $m \in [0,1]$ with strict inequality for some $m \in (0,1)$. Hence for any two distinct societies S and S' that fully lie on the same side of the equality line, we have $S \succ^D S'$ iff S' lies in between S and the equality line. It is immediate to see that $\succ^D$ is a strict partial order; it is asymmetric, transitive but not necessarily complete. It captures a sense of intensity of the stochastic dominance: Given $S \succ^D S'$, we deduce that the same group stochastically dominates the other in both societies, say $\mathbf{W} \succ^{SD} \mathbf{M}$. But we also deduce that the stochastic dominance is stronger in S since for any $\%x$ of men occupying the top of the Men distribution, there is always more women that are in an equal to or better position than those men in S than S'. The following proposition summarizes the relationship between the Dominance preorder and dominance weakening transfers.

Proposition 1 *For any two societies S and S', if one can reach from S to S' by a series of dominance weakening transfers, then we have $S \succ^D S'$.*

It is worthwhile to note that the converse is not true. One can find two societies such that one lies in between the other and the equality line, yet it is not possible to reach from one to the other with dominance weakening transfers. For instance, for the societies below although $f^S(m) \geq f^{S'}(m)$ for all $m \in [0,1]$ and $\mathbf{W} \succ^{SD} \mathbf{M}$ and $\mathbf{W}' \succ^{SD} \mathbf{M}'$, it is not possible to reach from S to S' by dominance weakening transfers. This is because $\mathbf{M} \succ^{SD} \mathbf{M}'$, hence moving from S to S' would also require demotion of Men which cannot be achieved by dominance weakening transfers.

$$S = \begin{pmatrix} 30 & 20 \\ 40 & 30 \\ 30 & 50 \end{pmatrix} \qquad S' = \begin{pmatrix} 30 & 20 \\ 20 & 20 \\ 50 & 60 \end{pmatrix}$$

Dominance preorder suggests a reasonable way to compare societies in terms of the inequality between groups. However it can only be used to evaluate very specific societies; societies with stochastic dominance between groups. One way to extend this partial comparison to the domain of all societies is to come up with indices that agree on the ranking of the Dominance preorder, yet are defined for all possible societies. That is what we do in the next section.

2 Extending the Partial Ranking: The Net Difference Index

Given the analogies so far between the Lorenz curve and the Domination curve, a natural extension of the Dominance preorder can be reached by mimicking the relationship between the Gini Inequality Index and the Lorenz curve. Gini Inequality Index is equal to the ratio of the area between the Lorenz curve and the equality line to the area under the equality line. One crucial difference between the Lorenz curve and the Domination curve is that, the latter can reach both above and below the equality line, inducing a sense of 'direction' to the inequality. Taking this into account, we compute 'the net area', the area between the curve and the equality line above the equality line minus the area between them below the equality line and we arrive at the Net Difference Index (Lieberson, 1976). Let us first formally define the index, before showing the relationship to the Domination Curve formally in Proposition 2:[10]

$$\begin{aligned} ND(S) &= \frac{\sum_i (W_i \sum_{i+1}^{n} M_j - M_i \sum_{i+1}^{n} W_j)}{WM} \\ &= \sum_i (w_i \sum_{i+1}^{n} m_j - m_i \sum_{i+1}^{n} m_j) \end{aligned}$$

Given a society S, the Net Difference Index, $ND(S)$, measures inequality in terms of the number of times a group ranks higher than the other group in pairwise confrontations. We define **a domination** by a group as having a member in a better position than a counter-group member. For instance, a woman in position i occupies a better position than all the men that are in worse positions than i, thus she creates $\sum_{i+1}^{n} M_j$ dominations in total. Then, ND is equal to the net difference in average number of dominations by Women and Men.

Intuitively, ND gives out the ex-ante probability advantage between groups: For a random pair of a woman and a man, the difference in probabilities of one individual being in a better position than the other.

ND is a directional measure. It takes values between -1 and 1, 0 being complete equality, 1 being maximum inequality advantaging Women and -1 being maximum inequality advantaging Men. ND respects the ordering suggested by the Dominance preorder, i.e., if $S \succ^D S'$ then $|ND(S)| > |ND(S')|$. Finally, Proposition 2 establishes the promised relation between the Dominance curve and ND:

Proposition 2 $ND(S) = \dfrac{\textit{Net area between the Dominance curve and the equality line}}{\textit{The area below the equality line}}$

The proof of Proposition 2 is merely based on the integration of f^S.

2.1 Characterizing Properties

A between group ordinal inequality measure is a continuous function $H : C \to \mathbb{R}$ that attaches to each possible society S, a real number indicating the amount of inequality between

[10] We abuse notation and use $\sum_i$ to denote $\sum_{i=1}^{n}$, $\sum_{i+1}^{n}$ to denote $\sum_{j=i+1}^{n}$ and so on.

the distributions of groups across ordered positions. In this subsection we list and discuss the properties on H characterizing the Net Difference Index. Let us start with a property that relates to the previous section:

Strong Transfers (ST): Let $S = (\mathbf{W}, \mathbf{M})$ and $S' = (\mathbf{W}', \mathbf{M}')$ such that one of the following holds:

(i) $\mathbf{W} \succ^{SD} \mathbf{M}$, $(\mathbf{W}')^T = (W_1, ..., W_i - \delta, ..., W_j + \delta, ..., W_n)$ for some $\delta > 0$ and $i < j$, $\mathbf{M} = \mathbf{M}'$ and $\mathbf{W}' \succ^{SD} \mathbf{M}'$

(ii) $\mathbf{W} \succ^{SD} \mathbf{M}$, $\mathbf{W} = \mathbf{W}'$ and $(\mathbf{M}')^T = (M_1, ..., M_i + \delta, ..., M_j - \delta, ..., M_n)$ for some $\delta > 0$ and $i < j$, and $\mathbf{W}' \succ^{SD} \mathbf{M}'$

(iii) $\mathbf{M} \succ^{SD} \mathbf{W}$, $(\mathbf{M}')^T = (M_1, ..., M_i - \delta, ..., M_j + \delta, ..., M_n)$ for some $\delta > 0$ and $i < j$, $\mathbf{W} = \mathbf{W}'$ and $\mathbf{M}' \succ^{SD} \mathbf{W}'$

(iv) $\mathbf{M} \succ^{SD} \mathbf{W}$, $\mathbf{M} = \mathbf{M}'$ and $(\mathbf{W}')^T = (W_1, ..., W_i + \delta, ..., W_j - \delta, ..., W_n)$ for some $\delta > 0$ and $i < j$ and $\mathbf{M}' \succ^{SD} \mathbf{W}'$.

Then, $|H(S')| \leq |H(S)|$. Moreover, if $M_j \neq 0$ for (i) and (iv) or $W_j \neq 0$ for (ii) and (iii), then $|H(S')| < |H(S)|$.

ST simply states that dominance weakening transfers cannot increase the amount of inequality. Moreover if a dominance weakening transfer is made to a position that is not null for the counter group, then inequality decreases.

ND is a measure that takes into account the direction of the inequality between groups. In Subsection 2.3, we discuss in depth the version of ND without the directionality property, but now, for characterization purposes we state directionality as a separate property. DR ensures that exchanging the distributions of Women and Men reverses the direction of the inequality. The argument for directionality is not too difficult to defend for two social group settings such as women vs men or white vs non-white origin; one would not only be interested in how the level of inequality changes over time and space but also whether inequality always favor the same social group or no.

Directionality (DR): For any $S = (\mathbf{W}, \mathbf{M})$, we have $H(\mathbf{W}, \mathbf{M}) = -H(\mathbf{M}, \mathbf{W})$.

The following is a property that we borrow from segregation literature and modify according to the ordinal information in our setting. Consider two societies S and S', that are equal to each other in all aspects but there is only one position in S' corresponding to two successive positions with equal women to men ratios in S'. Hence S has n positions, whereas

S' has $n-1$. Basically it is as if S' is obtained from S by combining two successive positions with the same group ratio. Successive Proportionate Merges ensures that the inequality between groups remain unchanged, i.e, combining two successive positions with the same women to men ratios do not change inequality.

Successive Proportionate Merges (SPM): Let S be a society over n positions such that $\exists\ k < n$ with $\frac{W_k}{M_k} = \frac{W_{k+1}}{M_{k+1}}$. Let S' be a society over $n' = (n-1)$ positions such that $W'_i = W_i$, $M'_i = M_i$ for $i = 1, ..., k-1$; $W'_k = W_k + W_{k+1}$, $M'_k = M_k + M_{k+1}$, and $W'_i = W_{i+1}$, $M'_i = M_{i+1}$ for $i = k+1, ..., n-1$. Then $H(S) = H(S')$.

SPM highlights when the ordinal information about the positions becomes idle. For two successive positions, the fact that one is better than the other is relevant for inequality only if the relative distributions of the social groups differ over these positions. Notice that combining two positions is not disregarding all ordinal information regarding these two positions, it is only disregarding the ordinal information *between* them: the individuals occupying these positions are still in better (worse) positions than all the other individuals they were jointly dominating (dominated by) before.

The following is a technical normalization property that sets the inequality equal to 0 for societies with only one non-empty position.

Normalization (NORM): For any S such that $\exists k \in \{1, ..., n\}$ with $m_k = 1$ and $w_k = 1$, then $H(S) = 0$.

NORM simply normalizes the group inequality to 0 for the societies that possess nothing to compare.

Decomposability is a crucial property for characterizations of inequality indices in the entire literature not only because it mathematically helps to pin down the family of indices but also it has practical implications. Decomposability shows how to aggregate inequalities in different subparts of the society consistently. Quite often empirical researches are interested in the concentration of inequality in various parts of the society such as geographical locations or within different subgroups such as ethnic groups. Decomposable indices allow us to express the overall inequality in the society as an aggregation of the inequalities in different subparts of the society.

Remembering the graphical representation of the Net Difference and its similarity to the relationship between Gini and the Lorenz curve, it is not immediately clear what kind of a decomposability property the Net Difference might satisfy.[11] Since the focus of our interest is the inequality between groups, a natural decomposition would be over different subgroups of the social groups, where a subgroup refers to a subset of a social group. For instance within the group of Women, two subgroups of interest could be Immigrant Women and Local Women. To formalize let us consider two subgroups of Women, say, $\mathbf{W}$ and $\mathbf{W}'$. The two subsocieties then would be $(\mathbf{W}, \mathbf{M})$ and $(\mathbf{W}', \mathbf{M})$ where the main society is $(\mathbf{W} + \mathbf{W}', \mathbf{M})$. A **decomposable** index allows the overall inequality between groups to be expressed as an aggregation of the inequalities in the subsocieties $(\mathbf{W}, \mathbf{M})$ and $(\mathbf{W}', \mathbf{M})$.[12] We define an **additively decomposable** index H as a function that allows the overall inequality in S to be expressed as a weighted sum of the inequalities in the subsocieties; i.e., $H(\mathbf{W} + \mathbf{W}', \mathbf{M}) = \alpha(W, W + W')H(\mathbf{W}, \mathbf{M}) + \alpha(W', W + W')H(\mathbf{W}', \mathbf{M})$, for some weight function $\alpha(\cdot)$, that depends on the number of subgroup members in the corresponding society as well as the number of group members in the overall society.

Certainly the weight function $\alpha(\cdot)$ takes different forms depending on the other properties satisfied by the index. Below we show that NORM and ST restricts the admissible class of weights functions to those that could be written as a ratio of a function of the number of subgroup members to the total number of group members.

Proposition 3 *If an additively decomposable index H satisfies NORM and ST, then for $X = W, M$, the subpopulation weights can be written as*

$$\alpha(X^i, X) = \frac{g(X^i)}{g(X)} \tag{1}$$

for some function $g(\cdot)$ that is nowhere equal to 0.

The weight function $\alpha(\cdot)$ for the decomposability of the Net Difference Index unsurprisingly takes the identity function as $g(\cdot)$, i.e., $\alpha(W, W + W') = \frac{W}{W+W'}$ and $\alpha(M, M + M') = \frac{M}{M+M'}$.

Decomposability (DEC): For any $S = (\mathbf{W} + \mathbf{W}', \mathbf{M})$, we have

$$H(S) = \frac{W}{W + W'} H(\mathbf{W}, \mathbf{M}) + \frac{W'}{W + W'} H(\mathbf{W}', \mathbf{M})$$

[11] The Gini Index does not belong to the group of additively decomposable income inequality indices. For more on decomposability of Gini, see Bourguignon (1979); Dagum (1998); Lambert and Aronson (1993).

[12] In principle, the subgroups of the other group, say $\mathbf{M}$ and $\mathbf{M}'$, can also be of interest. In that case, overall inequality will be equal to an aggregation of inequalities in $(\mathbf{W}, \mathbf{M})$, $(\mathbf{W}', \mathbf{M}')$, $(\mathbf{W}, \mathbf{M}')$ and $(\mathbf{W}', \mathbf{M})$.

and similarly for any $S = (\mathbf{W}, \mathbf{M} + \mathbf{M}')$ we have

$$H(S) = \frac{M}{M + M'} H(\mathbf{W}, \mathbf{M}) + \frac{M'}{M + M'} H(\mathbf{W}, \mathbf{M}').$$

We stated DEC for two subsocieties for simplicity, but certainly it implies DEC for more than two subsocieties.

We are now ready to introduce the main result of the paper. These properties listed not only are satisfied by ND, but also they do characterize it up to a scalar transformation. As we will show by Lemma 2 in the Proof of Theorem 1, DEC together with DR implies NORM, hence we drop NORM from the statement of the theorem.

Theorem 1 $H : C \to \mathbb{R}$ *satisfies ST, DR, SPM and DEC if and only if it is a scalar transformation of the Net Difference Index.*

We discuss independence and the implications of the characterizing properties in the next subsection.

2.2 Independence and Other Related Indices

All of the characterizing properties are independent. ST is the only property that eliminates a constant 0 function, i.e., $H(S) = 0$ satisfies all other properties but ST. DR not only assigns a direction to the measured inequality but does this in a symmetric way around 0. An index that evaluates dominations by Women and Men asymmetrically can be an example to a group inequality function that satisfies all of the other properties but DR. For instance,

$$H(\mathbf{W}, \mathbf{M}) = \sum_i (2w_i \sum_{i+1}^{n} m_j - m_i \sum_{i+1}^{n} w_j).$$

In the next subsection, we suggest and characterize a version of the index without directionality.

SPM highlights the noncardinality of the variable of interest. For two successive positions, the fact that one is better than the other is relevant for inequality only if the relative distributions of the social groups differ over these positions according to SPM. However, if there *is* actually more information regarding the ranking of the positions rather than pure ordinal information, one might consider to use a weighted version of the index:

$$ND^W(S) = \sum_i c_i (w_i \sum_{i+1}^{n} m_j - m_i \sum_{i+1}^{n} w_j)$$

where $c_i : \{1, 2, ..., n\} \to R$ is a weighting function or simply a cardinal scale. As long as $c_i(\cdot)$ is a strictly decreasing function, this index would satisfy all the other properties but SPM.

An example to a function that satisfies all properties but DEC would be the version of the index that evaluates the difference in total number of dominations rather than averages:

$$ND^A(S) = \sum_i (W_i \sum_{i+1}^{n} M_j - M_i \sum_{i+1}^{n} W_j).$$

It is easy to come up with scenarios where the actual number of individuals within the social groups matter as much as the distribution over positions. In that case, this absolute version of the index would serve to the purpose. ND^A fails DEC but satisfies all the other properties. However D^A is also a decomposable function; it satisfies an unweighted version of DEC, i.e., $ND^A(\mathbf{W} + \mathbf{W}', \mathbf{M}) = ND^A(\mathbf{W}, \mathbf{M}) + ND^A(\mathbf{W}', \mathbf{M})$, where the subsocieties are defined as before. Indeed ND^A is characterized by all the other properties in addition to this absolute decomposability property.

2.3 Symmetry instead of Directionality

Directionality of a between group inequality measure might be a useful property in settings with only two social groups and when the direction of inequality indeed matters for policy purposes. However it might not always be a desirable property for a practical, summary measure of inequality, especially for comparisons across societies with different social groups. Moreover once extension of the index to multi-group settings is considered, as we do in Section 3, directionality becomes burdensome. A very natural question becomes whether we can extend the Dominance preorder without directionality, and hence, we explore the absolute value of the difference in average number of dominations by Women and Men. Let us call this measure the Domination Index, D:

$$\begin{aligned} D(S) &= \left| \frac{\sum_i (W_i \sum_{i+1}^{n} M_j - M_i \sum_{i+1}^{n} W_j)}{WM} \right| \\ &= \left| \sum_i (w_i \sum_{i+1}^{n} m_j - m_i \sum_{i+1}^{n} w_j) \right| \end{aligned}$$

D takes values between 0 and 1, 0 being complete equality and 1 being maximum inequality. The characterization of the Domination Index certainly follows similar principles to that of the Net Difference Index with two crucial differences. First, we replace Directionality with a Symmetry property, ensuring that swapping the distributions of Women and Men does not change inequality:

Symmetry for Groups (SYM): For any $S = (\mathbf{W}, \mathbf{M})$, we have $H(\mathbf{W}, \mathbf{M}) = H(\mathbf{M}, \mathbf{W})$.

Second, and more critically, we need to modify Decomposability. This is because the Domination Index is NOT an additively decomposable index. To see why, consider S with 100 Women and 100 Men over 2 positions and its decomposition into two subsocieties as follows:

$$S = (\mathbf{W}, \mathbf{M}) = \begin{pmatrix} 50 & 50 \\ 50 & 50 \end{pmatrix} \to (\mathbf{W}^1, \mathbf{M}) = \begin{pmatrix} 50 & 50 \\ 0 & 50 \end{pmatrix} \text{ and } (\mathbf{W}^2, \mathbf{M}) = \begin{pmatrix} 0 & 50 \\ 50 & 50 \end{pmatrix}$$

Women and Men are distributed perfectly equally in S. For an index H that satisfies NORM and SPM, $H(S) = 0$. However neither $(\mathbf{W}^1, \mathbf{M})$ nor $(\mathbf{W}^2, \mathbf{M})$ are equal societies. For an index H that satisfies ST and takes only nonnegative values, we have $H(\mathbf{W}^1, \mathbf{M}) > 0$ as well as $H(\mathbf{W}^2, \mathbf{M}) > 0$. It is not possible to express the inequality in S as a weighted average of the inequalities in constituent subsocieties. That is why Domination Index is not additively decomposable. However for certain decompositions, where the inequality in subsocieties are in the same *direction*, the overall inequality can indeed be expressed as a weighted average of the inequalities in constituent subsocieties. Consider the following example:

$$D(S') = D\begin{pmatrix} 140 & 50 \\ 60 & 50 \end{pmatrix} = \frac{1}{2} D\begin{pmatrix} 80 & 50 \\ 20 & 50 \end{pmatrix} + \frac{1}{2} D\begin{pmatrix} 60 & 50 \\ 40 & 50 \end{pmatrix}$$
$$0.2 = \frac{1}{2} 0.3 + \frac{1}{2} 0.1$$

The main difference between the two examples above is the *direction* of inequality. In the decomposition of S, Women are more advantageous in the first subsociety, whereas are Men in the second one. When two subgroups are actually considered together in S, these advantages cancel out. However in the decomposition of S', Women are more advantageous than Men in both subsocieties; the between group inequality is favoring the same social group, hence there is no cancelling out when the entire group is considered. That is simply the intuition behind the decomposability of the Domination Index: D is additively decomposable with relative population weights as long as the inequalities in the subsocieties are *favoring the same group*. Certainly we need to quantify what is meant by 'favoring' a group. We classify

S as of **W-type**, if inequality is favoring Women as opposed to **M-type** if inequality is favoring Men. Formally;

We say that S is of **W-type** if $H(S) = 0$ or there exist two positions k, l with $k < l$ and a $\Delta > 0$ such that for $S = (\mathbf{W}, \mathbf{M})$ with $\mathbf{W} = (W_1, ..., W_k, ..., W_l, ...W_n)$, we have $H(S) > H(S')$ for any $S' = (\mathbf{W}', \mathbf{M})$ with $\mathbf{W}' = (W_1, ..., W_k - \varepsilon, ..., W_l + \varepsilon, ..., W_n)$ and $\varepsilon \leq \Delta$. We say that S is of **M-type** if S is not W-type or $H(S) = 0$. That is to say, S is of W-type if there exist positions $k < l$ and $\Delta > 0$ such that demoting at most Δ amount of women from k to l decreases inequality as measured by H. If one cannot find such positions or Δ as defined, then S is of M-type. If S is such that $H(S) = 0$, then S is both W-type and M-type.

The main intuition behind this index-dependent classification is that societies for which it is possible to decrease inequality (as measured by H) by demoting women would be the ones that are favoring Women initially; and societies that it is never possible to decrease inequality by demoting Women would be the ones that are favoring Men. Different indices will classify societies into different types since they would evaluate not only the level but also the direction of inequality in different ways. Any index satisfying Within-type Decomposability is decomposable for over same type subsocieties according to its own classification. It is worth noting that for any H satisfying ST, if in S, $\mathbf{W} \succ^{SD} \mathbf{M}$, then S has to be of W-type, immediate to see by definition of ST. Similarly, if $\mathbf{M} \succ^{SD} \mathbf{W}$ instead, then S is of M-type. Given H, if S is not of one type, then it has to be of the other by definition. Moreover, we have $H(S) = 0$ if and only if S is of both W-type and M-type.

Within-type Decomposability (T-DEC): For any $S = (\mathbf{W} + \mathbf{W}', \mathbf{M}) \in C$, we have

$$H(S) = \frac{W}{W + W'} H(\mathbf{W}, \mathbf{M}) + \frac{W'}{W + W'} H(\mathbf{W}', \mathbf{M})$$

as long as $(\mathbf{W}, \mathbf{M})$ and $(\mathbf{W}', \mathbf{M})$ are of the same type. Similarly for any $S = (\mathbf{W}, \mathbf{M} + \mathbf{M}') \in C$ we have

$$H(S) = \frac{M}{M + M'} H(\mathbf{W}, \mathbf{M}) + \frac{M'}{M + M'} H(\mathbf{W}, \mathbf{M}')$$

as long as $(\mathbf{W}, \mathbf{M})$ and $(\mathbf{W}, \mathbf{M}')$ are of the same type.

Theorem 2 characterizes D with ST, NORM, SYM properties from before, jointly with the newly introduced SYM and T-DEC.

Theorem 2 *$H : C \to \mathbb{R}_+$ satisfies ST, SYM, NORM, SPM and T-DEC if and only if it is a positive scalar transformation of the Domination Index.*

3 Concluding Remarks and Possible Extensions

Unequal distribution of social groups across different levels of welfare is quite commonly observed. When we go beyond income inequality and consider non-cardinal welfare determining variables such as education, health, occupation or subjective well-being, we run short of well-developed inequality measurement techniques. This paper aimed to analyze an intuitive and well-founded methodology to evaluate non-income inequalities between two social groups without appealing to additional cardinalisation assumptions. We conclude with two possible extensions.

A natural way to extend the measurement method analysed in this paper to settings with more than two groups is to consider an aggregation of the differences in pairwise dominations for each pair of groups. When there are more than two social groups, we first compute the average difference in number of dominations for each pair. The average of these average differences would be the multi-group Domination Index. Let us state this idea formally: Let $\mathcal{G}$ be a set of social groups with cardinality G. Then a society matrix S with G groups will be of dimension $n \times G$ and the multi-group Domination Index would be equal to $\frac{1}{2G}\sum_{M\in\mathcal{G}}\sum_{N\in\mathcal{G}} D(\mathbf{M},\mathbf{N})$, where $\mathbf{M}$ and $\mathbf{N}$ denote the distributions of groups M and N in S respectively. Notice this still captures the extra probability that on a random selection of a pair of individuals from different groups, the member of one group rank-dominates the other. The foundational analysis of this multi-group Domination Index requires further research.

Having focused our attention to ordinal inequalities, we assumed full comparability of the categories. A second extension can be suggested for only partially comparable categories. Consider the attributes of an occupation such as wage, prestige, working conditions, etc. An occupation may have quite challenging working conditions, even resulting in health troubles, although offering a very high level of wage. How this occupation would compare to one with better working conditions but lower pay is not obvious. Hence taking multiple attributes into account might result in only a partial ordering of occupations rather than a linear one. Similarly, consider a setting where two aspects of welfare are taken into account simultaneously in determining the positions, such as health and happiness. Both health and happiness data are examples to ordinal categorical data, however taking both of them into account at the same time would result in partial ordering of the positions (if we are to avoid extra assumptions such as having more health is better than having more happiness). Hence the question becomes how to compare distributions of groups over partially ordered categories. One suggestion we have is the Maximum Group Inequality index: Formally, let $P_{\mathcal{I}}$ be a strict partial order over a set of positions $\mathcal{I}$. A society will be a pair of elements $(S, P_{\mathcal{I}})$, where S is the usual society matrix. Let $\mathcal{L}^{P_{\mathcal{I}}}$ denote the set of linear extensions of P over $\mathcal{I}$, i.e.; the set of complete, transitive and asymmetric binary relations over $\mathcal{I}$ with for all $L_{\mathcal{I}}$ in $\mathcal{L}^{P_{\mathcal{I}}}$, iLj if iPj. Then, Maximum Group Inequality Index, M, will be: $M(S, P_{\mathcal{I}}) = \max_{L_{\mathcal{I}} \in \mathcal{L}^{P_{\mathcal{I}}}} D(S_{L_{\mathcal{I}}})$, where $S_{L_{\mathcal{I}}}$ refers to the society matrix with the linear order $L_{\mathcal{I}}$. As before, M takes values in $[0, 1]$. If there is no missing information about the ordering of the positions, M is equal to D. In case of some missing information, M gives the maximum possible level of group inequality, which refers to the worst-case scenario of the society. If two positions remain uncompared by the original ordering, this will be because of the fact that there is no unique universal way of ranking these positions. Considering the worst-case scenario is consistent with a Rawlsian framework of welfare. The algorithmic structure and behavior of the Maximum Group Inequality Index remain to be explored.

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A Appendix

Proof of Proposition 1 Consider S and S' as defined and assume that one can reach from S to S' by a series of dominance weakening transfers. Assume without loss of generality that $\mathbf{W} \succ^{SD} \mathbf{M}$. Then by definition, for any i, $\sum_1^i w_k \geq \sum_1^k m_k$ and $\sum_1^i w'_k \geq \sum_1^i m'_k$ with at least one strict inequality for each society. Moreover, since by demoting women and promoting men one can reach from S to S', we have for any i, $\sum_1^i w_k \geq \sum_1^i w'_k$ and $\sum_1^i m'_k \geq \sum_1^i m_k$ with at least one strict inequality. Combining these, we arrive at $\sum_1^i w_k \geq \sum_1^i w'_k \geq \sum_1^i m'_k \geq \sum_1^i m_k$, which directly implies that both curves lie over the equality line and S' lies in between S and the equality line as claimed. □

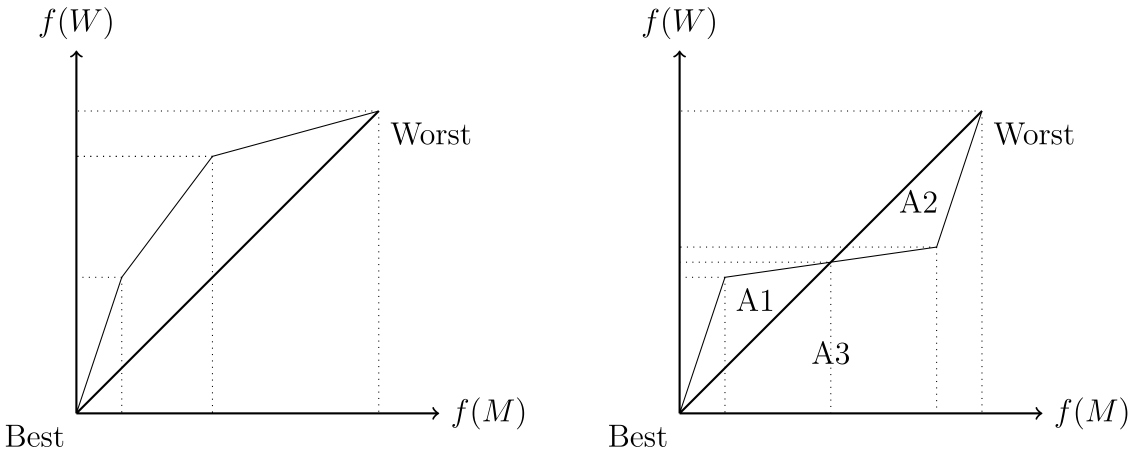
Proof of Proposition 2 The proof is by integration. First notice that for any society S with $n > 1$, the function described by the Dominance Curve, $f^S : [0,1] \to [0,1]$ can equivalently be expressed as

$$f^S(x) = \begin{cases} \frac{w_1}{m_1}x & \text{if } x \leq m_1 \\ \frac{w_2}{m_2}x + \frac{w_1 m_2 - w_2 m_1}{m_2} & \text{if } \quad m_1 \leq x \leq m_1 + m_2 \\ \ldots & \ldots \\ \frac{w_i}{m_i}x + \frac{m_i \sum_1^{i-1} w_j - w_i \sum_1^{i-1} m_j}{m_i} & \text{if } \sum_1^{i-1} m_j \leq x \leq \sum_1^i m_j \\ \ldots & \ldots \\ \frac{w_n}{m_n}x + \frac{m_n \sum_1^{n-1} w_j - w_i \sum_1^{n-1} m_j}{m_n} & \text{if } \sum_1^{n-1} m_j \leq x \leq 1 \end{cases}$$

Now consider a society S whose Domination curve lies fully above the equality line, such as the one in Figure 2a.

If $n = 1$, the Dominance Curve lies exactly on the equality line and $ND(S) = 0$. Let $n > 1$.

Figure 2: Net Difference Index and Dominance Curve



(a) (b)

The area A between the Dominance Curve and the equality line would be:

$$
\begin{aligned}
A =& \int_0^{m_1} \frac{w_1}{m_1} x dx + \sum_{i=2}^{n} \int_{\sum_1^{i-1} m_j}^{\sum_1^i m_j} \frac{w_i}{m_i} x + \frac{m_i \sum_1^{i-1} w_j - w_i \sum_1^{i-1} m_j}{m_i} dx - \int_0^m \frac{w}{m} x dx \\
=& \frac{w_1 m_1^2}{2m_1} + \sum_2^n \frac{w_i}{2m_i}[(\sum_1^i m_j)^2 - (\sum_1^{i-1} m_j)^2] + \sum_{i=1}^n \frac{m_i \sum_1^{i-1} w_j - w_i \sum_1^{i-1} m_j}{m_i} m_i - \frac{wm}{2} \\
=& \frac{1}{2}[\sum_{i=1}^n w_i(\sum_1^i m_j + \sum_1^{i-1} m_j) - wm] + \sum_{i=1}^n (m_i \sum_1^{i-1} w_j - w_i \sum_1^{i-1} m_j) \\
=& \frac{1}{2}(\sum_i^n w_i \sum_1^i m_j + \sum_1^n w_i \sum_1^{i-1} m_j - \sum_1^n w_i \sum_1^n m_j) + \sum_{i=1}^n (m_i \sum_1^{i-1} w_j - w_i \sum_1^{i-1} m_j) \\
=& \frac{1}{2} \sum_{i=1}^n (m_i \sum_1^{i-1} w_j - w_i \sum_1^{i-1} m_j) = \frac{1}{2} \sum_{i=1}^n (w_i \sum_{i+1}^n m_j - m_i \sum_{i+1}^n w_j) \\
=& \frac{1}{2} ND(S)
\end{aligned}
$$

Since the area below the equality line is equal to $\frac{1}{2}$, we prove the claim for S. Now consider any S, whose dominance curve might also lie below the equality line, like the one in Figure 2b. The proof above also applies in this case. To see that consider the net area in Figure 2b: $A1 - A2 = \int_0^1 f^S(x).dx - \int_0^1 x dx = (A1 + A3) - (A2 + A3)$, establishing the proof. $\square$.

Proof of Proposition 3 Take any S and let H be as defined. First consider the following decomposition into subsocieties

$$H(S) = \alpha(W_1, W)H \begin{pmatrix} W_1 & M_1 \\ 0 & M_2 \\ 0 & M_3 \\ \cdots & \cdots \\ 0 & M_n \end{pmatrix} + \alpha(W_2, W) \begin{pmatrix} 0 & M_1 \\ W_2 & M_2 \\ 0 & M_3 \\ \cdots & \cdots \\ 0 & M_n \end{pmatrix} + \alpha(\sum_3^n W_i, W) \begin{pmatrix} 0 & M_1 \\ 0 & M_2 \\ W_3 & M_3 \\ \cdots & \cdots \\ W_n & M_n \end{pmatrix}. \tag{2}$$

Now consider the following decomposition

$$\begin{aligned} H(S) =& \alpha(\sum_1^2 W_i, W)H \begin{pmatrix} W_1 & M_1 \\ W_2 & M_2 \\ 0 & M_3 \\ \cdots & \cdots \\ 0 & M_n \end{pmatrix} + \alpha(\sum_3^n W_i, W) \begin{pmatrix} 0 & M_1 \\ 0 & M_2 \\ W_3 & M_3 \\ \cdots & \cdots \\ W_n & M_n \end{pmatrix} \\ =& \alpha(\sum_1^2 W_i, W)\alpha(W_1, \sum_1^2 W_i)H \begin{pmatrix} W_1 & M_1 \\ 0 & M_2 \\ 0 & M_3 \\ \cdots & \cdots \\ 0 & M_n \end{pmatrix} + \alpha(\sum_1^2 W_i, W)\alpha(W_2, \sum_1^2 W_i) \begin{pmatrix} 0 & M_1 \\ W_2 & M_2 \\ 0 & M_3 \\ \cdots & \cdots \\ 0 & M_n \end{pmatrix} \\ &+ \alpha(\sum_3^n W_i, W) \begin{pmatrix} 0 & M_1 \\ 0 & M_2 \\ W_3 & M_3 \\ \cdots & \cdots \\ W_n & M_n \end{pmatrix}. \end{aligned} \tag{3}$$

Since $H(S)$ in (2) and (3) is the same, when we subtract (3) from (2), there remains

$$[\alpha(W_1, W) - \alpha(\sum_1^2 W_i, W)\alpha(W_1, \sum_1^2 W_i)]H \begin{pmatrix} W_1 & M_1 \\ 0 & M_2 \\ 0 & M_3 \\ \cdots & \cdots \\ 0 & M_n \end{pmatrix}$$

$$+ [\alpha(W_2, W) - \alpha(\sum_1^2 W_i, W)\alpha(W_2, \sum_1^2 W_i)] \begin{pmatrix} 0 & M_1 \\ W_2 & M_2 \\ 0 & M_3 \\ \cdots & \cdots \\ 0 & M_n \end{pmatrix} = 0. \qquad (4)$$

Since $\alpha(\cdot)$ is independent of $\mathbf{M}$, and since this equality has to hold for all S, consider $\mathbf{M}$ be such that $M_1 = W_1$ and $M_i = 0$ for all $i > 1$. In this case, the inequality measured by H in the first subsociety in 4 will be equal to 0 by NORM. Notice that the inequality measured by H in the second one cannot be equal to 0 by ST. Hence we have

$$\alpha(W_2, W) - \alpha(W_2, \sum_1^2 W_i)\alpha(\sum_1^2 W_i, W) = 0$$

or in general:

$$\alpha(W_i, W) - \alpha(W_i, W_i + W_j)\alpha(W_i + W_j, W) = 0.$$

Solving the functional equation $\alpha(x, y) = \alpha(x, z)\alpha(z, y)$, we arrive at

$$\alpha(W_i, W) = \frac{g(W_i)}{g(W)}$$

for some function $g(\cdot)$ that is nowhere 0. Notice that the initial decomposition into subgroups is without loss of generality. For any subgroup $\mathbf{W}^i$ of $\mathbf{W}$, we can first decompose $\mathbf{W}$ into two subgroups as follows: $\mathbf{W}_i$ plus Δ women from some other position and the rest of Women. In a second stage allocating $\mathbf{W}^i$ and Δ women to different subgroups would allow us replicate the steps above, establishing the general statement that

$$\alpha(W^i, W) = \frac{g(W^i)}{g(W)}$$

as claimed. Since additive decomposability is symmetric in social groups, the same holds for a decomposition over subgroups of Men. □

Proof of Theorem 1: We omit the proof of necessity part. To prove the sufficiency part, first we introduce a couple of lemmas.

Lemma 1 *Let $S = (\mathbf{W}, \mathbf{M})$ and $S' = (\alpha\mathbf{W}, \beta\mathbf{M})$ for some $\alpha, \beta > 0$. DEC implies that $H(S) = H(S')$.*

Proof: Consider any H satisfying DEC and any society $S = (\mathbf{W}, \mathbf{M})$. (i) Let $\alpha \in \mathbb{N}_{++}$. By using induction, we show that $H(\alpha\mathbf{W}, \mathbf{M}) = H(\mathbf{W}, \mathbf{M})$. For $\alpha = 2$, DEC implies $H(\mathbf{W}, \mathbf{M}) = \frac{1}{2}H(\mathbf{W}, \mathbf{M}) + \frac{1}{2}H(\mathbf{W}, \mathbf{M}) = H(2\mathbf{W}, \mathbf{M})$. Now assume the statement holds for $(\alpha - 1)$, i.e.: $H((\alpha-1)\mathbf{W}, \mathbf{M}) = H(\mathbf{W}, \mathbf{M})$. By DEC, $\frac{\alpha-1}{\alpha}H((\alpha-1)\mathbf{W}, \mathbf{M}) + \frac{1}{\alpha}H(\mathbf{W}, \mathbf{M}) = H(\alpha\mathbf{W}, \mathbf{M})$. But then, by the inductive argument: $\frac{\alpha-1}{\alpha}H(\mathbf{W}, \mathbf{M}) + \frac{1}{\alpha}H(\mathbf{W}, \mathbf{M}) = H(\mathbf{W}, \mathbf{M}) = H(\alpha\mathbf{W}, \mathbf{M})$ as claimed. (ii) Now consider $\alpha \in \mathbb{Q}_{++}$. Let $\alpha = \frac{p}{q}$ for some $p, q \in \mathbb{N}_{++}$. Then, repeated application of DEC ensures: $q\frac{p/q}{p}H(\frac{p}{q}\mathbf{W}, \mathbf{M}) = H(p\mathbf{W}, \mathbf{M})$. Since for $p \in \mathbb{N}_{++}$ we have $H(p\mathbf{W}, \mathbf{M}) = H(\mathbf{W}, \mathbf{M})$, we arrive; $H(\frac{p}{q}\mathbf{W}, \mathbf{M}) = H(\mathbf{W}, \mathbf{M})$ as claimed. (iii) Finally let $\alpha \in \mathbb{R}_{++}$. Since every irrational number can be expressed as the limit value of a sequence of rational numbers, let $\alpha = \lim q_i$ for some $q_i \in \mathbb{Q}_{++}$ $\forall i$. Then, $H(\alpha\mathbf{W}, \mathbf{M}) = H(\lim q_i\mathbf{W}, \mathbf{M}) = \lim H(q_i\mathbf{W}, \mathbf{M})$ by continuity of H. Since we have already showed that for any rational α the statement holds, we have: $H(\alpha\mathbf{W}, \mathbf{M}) = H(\mathbf{W}, \mathbf{M})$. The same argumentation over decomposition for $H(\mathbf{W}, \beta\mathbf{M})$ establishes the proof. □

Lemma 2 *DEC and DR implies NORM.*

Proof: Take S such that $\exists k \in \{1, ..., n\}$ such that $m_k = 1$ and $w_k = 1$. Then by Lemma 1 $H(\mathbf{W}, \mathbf{M}) = H(\frac{1}{W}\mathbf{W}, \frac{1}{M}\mathbf{M}) = H(\mathbf{w}, \mathbf{m})$, where $\mathbf{w} = \mathbf{m}$. But then by DR, $H(\mathbf{w}, \mathbf{m}) = -H(\mathbf{m}, \mathbf{w}) = 0$. □

Now we start with the proof of sufficiency. Let H be as defined and take any $S \in C$ such that for all i, $W_i \neq 0 \neq M_i$. If $n = 1$, then by NORM, $H(S) = 0$. Let $n \geq 2$. Denote $r_i = \frac{W_i}{M_i}$ for all i. The proof is mainly based on decomposition of S into elementary subsocities. However the decomposition applied depends on the ranking of r_i and r_{i+1} for each i.

First, let $r_1 \geq r_2$ and consider the following decomposition of $H(S)$

$$H\begin{pmatrix} W_1 & M_1 \\ W_2 & M_2 \\ ... & ... \\ W_n & M_n \end{pmatrix} = \frac{W_1'}{W} H\begin{pmatrix} W_1' & M_1 \\ 0 & M_2 \\ ... & ... \\ 0 & M_n \end{pmatrix} + \frac{W - W_1'}{W} H\begin{pmatrix} W_1 - W_1' & M_1 \\ W_2 & M_2 \\ ... & ... \\ W_n & M_n \end{pmatrix} \quad (5)$$

where W_1' is such that,

$$\frac{W_1 - W_1'}{M_1} = \frac{W_2}{M_2} = r_2$$
$$W_1'M_2 = W_1M_2 - M_1W_2.$$

Repeated application of SPM for the first subsociety in (5), as $W_j/M_j = 0$ for all $j > 1$, and again, SPM for the second subsociety yields

$$H(S) = \frac{W_1'}{W}\ H\begin{pmatrix} W_1' & M_1 \\ 0 & \sum_2^n M_j \end{pmatrix} + \frac{W - W_1'}{W}\ H\begin{pmatrix} W_1 + W_2 - W_1' & M_1 + M_2 \\ W_3 & M_3 \\ ... & ... \\ W_n & M_n \end{pmatrix}. \tag{6}$$

Finally, we decompose the first subsociety in (6) once more, yielding: $H(S) =$

$$\frac{W_1'}{W}\left[\frac{\sum_2^n M_j}{M} H\begin{pmatrix} W_1' & 0 \\ 0 & \sum_2^n M_j \end{pmatrix} + \frac{M_1}{M} H\begin{pmatrix} W_1' & M_1 \\ 0 & 0 \end{pmatrix}\right] + \frac{W - W_1'}{W}\ H\begin{pmatrix} W_1 + W_2 - W_1' & M_1 + M_2 \\ W_3 & M_3 \\ ... & ... \\ W_n & M_n \end{pmatrix}. \tag{7}$$

Since by Lemma 1, $H\left(\begin{smallmatrix} W_1' & 0 \\ 0 & \sum_2^n M_j \end{smallmatrix}\right) = H\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and by NORM, $H\left(\begin{smallmatrix} W_1' & M_1 \\ 0 & 0 \end{smallmatrix}\right) = 0$, we arrive at

$$H(S) = \frac{W_1'}{W}\frac{\sum_2^n M_j}{M}\ H\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{W - W_1'}{W}\ H\begin{pmatrix} W_1 + W_2 - W_1' & M_1 + M_2 \\ W_3 & M_3 \\ ... & ... \\ W_n & M_n \end{pmatrix}. \tag{8}$$

Next we decompose the second subsociety in (8) in a similar manner. There are two possible cases: In S, either $r_2 \geq r_3$ or $r_2 < r_3$. The decomposition we apply depends on the ordering of r_2 and r_3 in S.

Case 1: Let $r_2 \geq r_3$. We have $\frac{(W_1+W_2-W_1')}{M_1+M_2} \geq \frac{W_3}{M_3}$. Then, $H\left(\begin{smallmatrix} W_1+W_2-W_1' & M_1+M_2 \\ \vdots & \vdots \\ W_n & M_n \end{smallmatrix}\right) =$

$$= \frac{W_2'}{W - W_1'}\ H\begin{pmatrix} W_2' & M_1 + M_2 \\ 0 & M_3 \\ ... & ... \\ 0 & M_n \end{pmatrix} + \frac{W - \sum_1^2 W_i'}{W - W_1'}\ H\begin{pmatrix} \sum_1^2 W_i - \sum_1^2 W_i' & M_1 + M_2 \\ W_3 & M_3 \\ ... & ... \\ W_n & M_n \end{pmatrix} \tag{9}$$

where W_2' is such that

$$\frac{\sum_1^2 W_i - \sum_1^2 W_i'}{M_1 + M_2} = \frac{W_3}{M_3} = r_3. \tag{10}$$

SPM, NORM and further decomposition of the first subsociety in (9) similar to the one in (7), and substitution into (8) results in

$$\begin{aligned} H(S) =& \frac{W_1'}{W}\frac{\sum_2^n M_j}{M} H\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{W - W_1'}{W}\left[\frac{W_2'}{W - W_1'}\frac{\sum_3^n M_j}{M} H\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \\ &+ \frac{W - W_1'}{W}\frac{W - \sum_1^2 W_i'}{W - W_1'} H\begin{pmatrix} \sum_1^3 W_j - \sum_1^2 W_j' & \sum_1^3 M_j \\ W_4 & M_4 \\ \ldots & \ldots \\ W_n & M_n \end{pmatrix}. \end{aligned} \tag{11}$$

Finally, simplification yields

$$\begin{aligned} H(S) =& \frac{W_1'}{W}\frac{\sum_2^n M_j}{M} H\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{W_2'}{W}\frac{\sum_3^n M_j}{M} H\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &+ \frac{W - \sum_1^2 W_i'}{W} H\begin{pmatrix} \sum_1^3 W_j - \sum_1^2 W_j' & \sum_1^3 M_j \\ W_4 & M_4 \\ \ldots & \ldots \\ W_n & M_n \end{pmatrix}. \end{aligned} \tag{12}$$

Case 2: Let $r_2 < r_3$. We have $\frac{(W_1+W_2-W_1')}{M_1+M_2} < \frac{W_3}{M_3}$. Then, $H\left(\begin{smallmatrix} W_1+W_2-W_1' & M_1+M_2 \\ \ldots & \ldots \\ W_n & M_n \end{smallmatrix}\right) =$

$$= \frac{M_2'}{M} H\begin{pmatrix} W_1 + W_2 - W_1' & M_2' \\ W_3 & 0 \\ \ldots & \ldots \\ W_n & 0 \end{pmatrix} + \frac{M - M_2'}{M} H\begin{pmatrix} W_1 + W_2 - W_1' & M_1 + M_2 - M_2' \\ W_3 & M_3 \\ \ldots & \ldots \\ W_n & M_n \end{pmatrix} \tag{13}$$

where M_2' is such that

$$\frac{\sum_1^2 W_i - W_1'}{M_1 + M_2 - M_2'} = \frac{W_3}{M_3} = r_3. \tag{14}$$

Repeated application of SPM for the first subsociety in (13) and again, SPM for the second subsociety yields $H\left(\begin{smallmatrix} W_1+W_2-W_1' & M_1+M_2 \\ \dots & \dots \\ W_n & M_n \end{smallmatrix}\right) =$

$$= \frac{M_2'}{M}\ H\begin{pmatrix} \sum_1^2 W_i - W_1' & M_2' \\ \sum_3^n W_i & 0 \end{pmatrix} + \frac{M-M_2'}{M}\ H\begin{pmatrix} \sum_1^3 W_i - W_1' & \sum_1^3 M_j - M_2' \\ W_4 & M_4 \\ \dots & \dots \\ W_n & M_n \end{pmatrix}. \tag{15}$$

Finally, we decompose the first subsociety in (15) once more, yielding $H\left(\begin{smallmatrix} \sum_1^2 W_i-W_1' & M_1+M_2 \\ \dots & \dots \\ W_n & M_n \end{smallmatrix}\right) =$

$$\begin{aligned} =&\frac{M_2'}{M}\left[\frac{\sum_3^n W_i}{W-W_1'} H\begin{pmatrix} 0 & M_2' \\ \sum_3^n W_i & 0 \end{pmatrix} + \frac{\sum_1^2 W_i - W_1'}{W-W_1'} H\begin{pmatrix} \sum_1^2 W_i - W_1' & M_2' \\ 0 & 0 \end{pmatrix}\right] \\ &+ \frac{M-M_2'}{M}\ H\begin{pmatrix} \sum_1^3 W_i - W_1' & \sum_1^3 M_j - M_2' \\ W_4 & M_4 \\ \dots & \dots \\ W_n & M_n \end{pmatrix}. \end{aligned} \tag{16}$$

Since by Lemma 1, $H\left(\begin{smallmatrix} 0 & M_2' \\ \sum_3^n W_i & 0 \end{smallmatrix}\right) = H\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and by NORM, $H\left(\begin{smallmatrix} \sum_1^2 W_i-W_1' & M_2' \\ 0 & 0 \end{smallmatrix}\right) = 0$, we arrive at $H\left(\begin{smallmatrix} \sum_1^2 W_i-W_1' & M_1+M_2 \\ \dots & \dots \\ W_n & M_n \end{smallmatrix}\right) =$

$$=\frac{M_2'}{M}\ \frac{\sum_3^n W_i}{W-W_1'} H\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{M-M_2'}{M}\ H\begin{pmatrix} \sum_1^3 W_i - W_1' & \sum_1^3 M_j - M_2' \\ W_4 & M_4 \\ \dots & \dots \\ W_n & M_n \end{pmatrix}. \tag{17}$$

Plugging (17) into (8) results in

$$\begin{aligned} H(S) =&\frac{W_1'}{W}\frac{\sum_2^n M_j}{M} H\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sum_3^n W_i}{W}\ \frac{M_2'}{M}\ \ H\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &+ \frac{W-W_1'}{W}\frac{M-M_2'}{M}\ H\begin{pmatrix} \sum_1^3 W_i - W_1' & \sum_1^3 M_j - M_2' \\ W_4 & M_4 \\ \dots & \dots \\ W_n & M_n \end{pmatrix}. \end{aligned} \tag{18}$$

A comparison of equations (12) and (18) makes it clear that for all $i < n$ with $r_i \geq r_{i+1}$, the contribution of decomposition of row i as demonstrated to $H(S)$ is equal to

$$\frac{W_i' \sum_{i+1}^n M_j}{WM} H\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

whereas for all $i < n$ with $r_i < r_{i+1}$, the contribution to $H(S)$ is

$$\frac{M_i' \sum_{i+1}^n W_j}{WM} H \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where W_i' and M_i' is suitably defined so that SPM can be applied after the decomposition, as in equations (10) and (14). Thus, repetition of appropriate decompositions $(n-2)$ times results in:

$$\begin{aligned} H(S) = &\sum_{r_i \geq r_{i+1}}^{n-1} \frac{W_i'}{W} \frac{\sum_{i+1}^n M_j}{M} H \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{r_i < r_{i+1}}^{n-1} \frac{M_i'}{M} \frac{\sum_{i+1}^n W_j}{W} \quad H \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &+ \frac{W - \sum\limits_{r_i \geq r_{i+1}}^{n-1} W_i'}{W} \frac{M - \sum\limits_{r_i < r_{i+1}}^{n-1} M_i'}{M} H \begin{pmatrix} \left[\sum\limits_1^{n-1} W_i - \sum\limits_{r_i \geq r_{i+1}}^{n-1} W_1' \right] & \left[\sum\limits_1^{n-1} M_j - \sum\limits_{r_i < r_{i+1}}^{n-1} M_2' \right] \\ W_n & M_n \end{pmatrix} \end{aligned} \tag{19}$$

Now, first, let $r_{n-1} \geq r_n$. Then further decomposition of the third subsociety in (19) in conjunction with SPM, NORM and Lemma 1 yields

$$\begin{aligned} H(S) = &\sum_{r_i \geq r_{i+1}}^{n-1} \frac{W_i'}{W} \frac{\sum_{i+1}^n M_j}{M} H \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{r_i < r_{i+1}}^{n-1} \frac{M_i'}{M} \frac{\sum_{i+1}^n W_j}{W} \quad H \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &+ \frac{W - \sum\limits_{r_i \geq r_{i+1}}^{n-1} W_i'}{W} \frac{M - \sum\limits_{r_i < r_{i+1}}^{n-1} M_i'}{M} \frac{W_n}{W - \sum\limits_{r_i \geq r_{i+1}}^{n-1} W_i'} \frac{M_n}{M - \sum\limits_{r_i < r_{i+1}}^{n-1} M_i'} H \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ = &\sum_{r_i \geq r_{i+1}}^{n-1} \frac{W_i'}{W} \frac{\sum_{i+1}^n M_j}{M} H \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{r_i < r_{i+1}}^{n-1} \frac{M_i'}{M} \frac{\sum_{i+1}^n W_j}{W} \quad H \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &+ \frac{W_n'}{W} \frac{M_n}{M} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ = &\sum_{r_i \geq r_{i+1}}^{n} \frac{W_i'}{W} \frac{\sum_{i+1}^n M_j}{M} H \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{r_i < r_{i+1}}^{n} \frac{M_i'}{M} \frac{\sum_{i+1}^n W_j}{W} \quad H \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{20}$$

It is immediate to see that the other case with $r_{n-1} < r_n$ also implies equation (20). By DR, we have

$$H\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = K \Rightarrow H\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = -K$$

for some $K \in \mathcal{R}$. NORM and ST imply that $K \neq 0$. Thus, equation (20) becomes

$$H(S) = \left[\sum_{r_i \geq r_{i+1}}^{n} \frac{W_i'}{W}\frac{\sum_{i+1}^{n} M_j}{M} - \sum_{r_i < r_{i+1}}^{n} \frac{M_i'}{M}\frac{\sum_{i+1}^{n} W_j}{W}\right] K \tag{21}$$

As demonstrated by equations (10) and (14), W_i' and M_i' are chosen such that for all $i < n$:

$$\frac{\sum_1^i W_j - \sum_{r_j \geq r_{j+1}}^{i} W_j'}{\sum_1^i M_j - \sum_{r_j < r_{j+1}}^{i} M_j'} = \frac{W_{i+1}}{M_{i+1}}$$

$$M_{i+1}\sum_{r_j \geq r_{j+1}}^{i} W_j' - W_{i+1}\sum_{r_j < r_{j+1}}^{i} M_j' = M_{i+1}\sum_1^i W_j - W_{i+1}\sum_1^i M_j. \tag{22}$$

Summation of equation (22) over all $i < n$ finally gives

$$\sum_1^{n-1}\Big(M_{i+1}\sum_{r_j \geq r_{j+1}}^{i} W_j' - W_{i+1}\sum_{r_j < r_{j+1}}^{i} M_j'\Big) = \sum_1^{n-1}\Big(M_{i+1}\sum_1^i W_j - W_{i+1}\sum_1^i M_j\Big)$$

$$\sum_{r_i \geq r_{i+1}}^{n} W_i'\sum_{i+1}^{n} M_j - \sum_{r_i < r_{i+1}}^{n} M_i'\sum_{i+1}^{n} W_j = \sum_1^n W_i\sum_{i+1}^{n} M_j - \sum_1^n M_i\sum_{i+1}^{n} W_j.$$

Substitution into equation (21) yields

$$H(S) = \left[\sum_1^n\left(\frac{W_i}{W}\frac{\sum_{i+1}^{n} M_j}{M} - \frac{M_i}{M}\frac{\sum_{i+1}^{n} W_j}{W}\right)\right] K$$

establishing the functional form of H for any $S \in C$ with $W_i \neq 0 \neq M_i$ for all i. Now take any $S \in C$ and consider S' such that for all i with $W_i \neq 0 \neq M_i$, $W_i' = W_i$ and $M_i' = M_i$;

for all i with $W_i = 0$, $W'_i = \varepsilon$ and for all i with $M_i = 0$, $M'_i = \varepsilon$ for some ε in a small neighborhood of 0. By continuity of H

$$\begin{aligned}H(S) &= \lim_{\varepsilon\to 0} H(S') \\ &= \lim_{\varepsilon\to 0}\left[\sum_1^n \left(\frac{W'_i}{W'}\frac{\sum_{i+1}^n M'_j}{M'} - \frac{M'_i}{M'}\frac{\sum_{i+1}^n W'_j}{W'}\right)\right] K \\ &= \left[\sum_1^n \left(\frac{W_i}{W}\frac{\sum_{i+1}^n M_j}{M} - \frac{M_i}{M}\frac{\sum_{i+1}^n W_j}{W}\right)\right] K.\end{aligned}$$

□

Proof of Theorem 2: We omit the proof of necessity. To prove sufficiency, we introduce a few lemmas.

Lemma 3 *Let $S = (\mathbf{W}, \mathbf{M})$ and $S' = (\alpha\mathbf{W}, \beta\mathbf{M})$ for some $\alpha, \beta > 0$. T-DEC implies that $H(S) = H(S')$.*

Proof: The proof is an immediate implication of the proof of Lemma 1 of Theorem 1 and the fact that all subsocieties considered are of the same type. □

Lemma 4 *Let S be such that $\mathbf{W} \succ^{SD} \mathbf{M}$. ST implies that S is of W-type.*

Proof: Immediate by definition of stochastic dominance and ST. □

Lemma 5 *Take any S and S' such that $S = (\mathbf{W}, \mathbf{M})$ and $S' = (\alpha\mathbf{W}, \beta\mathbf{M})$ for some $\alpha, \beta > 0$. If S is of W-type (M-type), so is S'.*

Proof: The proof is immediate by definition of W-type (M-type): If we can find k, l, Δ as required for S, then k, l and $\alpha\Delta$ would suffice to show that S' is also of W-type or not. □

Now we start with the proof of Theorem 2. Let H be as defined and take any $S \in C$. If $n = 1$, then by NORM, $H(S) = 0$. Let $n \geq 2$.

In Step 1, we consider a very specific type of society and derive the functional form of H for it. Define $S \in C$ as **W-perfect** if $W_i > M_i > 0$ for all i and $\infty > r_1 > r_2 > \cdots > r_n > 0$, where $r_i = \frac{W_i}{M_i}$.

Step 1: Take $\mathbf{S} \in C$ that is W-perfect: We first show that S is of W-type. By Lemma 3, $H(\mathbf{W}, \mathbf{M}) = H(\mathbf{w}, \mathbf{m})$. By definition of W-perfection, for any $k = 1, \ldots, n-1$ and any $j = k+1, \ldots, n$, we have

$$\begin{aligned}
W_k M_j &> W_j M_k \\
\frac{W_k M_j}{WM} &> \frac{W_j M_k}{WM} \\
w_k m_j &> w_j m_k \\
w_k \sum_{k+1}^{n} m_j &> m_k \sum_{k+1}^{n} w_j.
\end{aligned} \tag{23}$$

Since this holds for each $k = 1, \ldots, n-1$, summing over all k

$$\begin{aligned}
\sum_{1}^{k} (w_i \sum_{k+1}^{n} m_j) &> \sum_{1}^{k} (m_i \sum_{k+1}^{n} w_j) \\
\sum_{1}^{k} w_i (1 - \sum_{1}^{k} m_j) &> \sum_{1}^{k} m_i (1 - \sum_{1}^{k} w_j) \\
\sum_{1}^{k} w_i - \sum_{1}^{k} w_i \sum_{1}^{k} m_j &> \sum_{1}^{k} m_i - \sum_{1}^{k} m_i \sum_{1}^{k} w_j \\
\sum_{1}^{k} w_i &> \sum_{1}^{k} m_i.
\end{aligned} \tag{24}$$

Hence $\mathbf{W} \succ^{SD} \mathbf{M}$. Then by Lemma 4, S is of W-type. Hence any W-perfect S is of W-type. Next, we start with the decomposition of S as in the Proof of Theorem 1. W-perfectness of S will allow the constituent subsocieties to be of W-type.

As $r_1 > r_2$, T-DEC implies

$$H \begin{pmatrix} W_1 & M_1 \\ W_2 & M_2 \\ \ldots & \ldots \\ W_n & M_n \end{pmatrix} = \frac{W_1'}{W} \; H \begin{pmatrix} W_1' & M_1 \\ 0 & M_2 \\ \ldots & \ldots \\ 0 & M_n \end{pmatrix} + \frac{W - W_1'}{W} \; H \begin{pmatrix} W_1 - W_1' & M_1 \\ W_2 & M_2 \\ \ldots & \ldots \\ W_n & M_n \end{pmatrix} \tag{25}$$

where W_1' is such that,

$$\frac{W_1 - W_1'}{M_1} = \frac{W_2}{M_2} = r_2.$$

Notice that the first subsociety in (25) is of W-type by stochastic dominance and Lemma 4 whereas the second subsociety is of W-type by the argument in equations (23), (24), stochastic dominance and Lemma 4. Hence T-DEC is applicable. Repeated application of SPM for the first subsociety in (25), as $W_j/M_j = 0$ for all $j > 1$, and again, SPM for the second subsociety yields

$$H(S) = \frac{W_1'}{W}\, H\begin{pmatrix} W_1' & M_1 \\ 0 & \sum_2^n M_j \end{pmatrix} + \frac{W - W_1'}{W}\, H\begin{pmatrix} W_1 + W_2 - W_1' & M_1 + M_2 \\ W_3 & M_3 \\ \dots & \dots \\ W_n & M_n \end{pmatrix}. \tag{26}$$

Finally, we decompose the first subsociety in (26) once more, yielding $H(S) =$

$$\frac{W_1'}{W}\left[\frac{\sum_2^n M_j}{M}\, H\begin{pmatrix} W_1' & 0 \\ 0 & \sum_2^n M_j \end{pmatrix} + \frac{M_1}{M}\, H\begin{pmatrix} W_1' & M_1 \\ 0 & 0 \end{pmatrix}\right] + \frac{W - W_1'}{W}\, H\begin{pmatrix} W_1 + W_2 - W_1' & M_1 + M_2 \\ W_3 & M_3 \\ \dots & \dots \\ W_n & M_n \end{pmatrix}.$$

Notice that T-DEC is applicable, since once again by Lemma 4, $\left(\begin{smallmatrix} W_1' & 0 \\ 0 & \sum_2^n M_j \end{smallmatrix}\right)$ is of W-type and by NORM, $H\left(\begin{smallmatrix} W_1' & M_1 \\ 0 & 0 \end{smallmatrix}\right) = 0$ and hence it is also of W-type. By Lemma 3, $H\left(\begin{smallmatrix} W_1' & 0 \\ 0 & \sum_2^n M_j \end{smallmatrix}\right) = H\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, yielding

$$H(S) = \frac{W_1'}{W}\frac{\sum_2^n M_j}{M}\, H\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{W - W_1'}{W}\, H\begin{pmatrix} W_1 + W_2 - W_1' & M_1 + M_2 \\ W_3 & M_3 \\ \dots & \dots \\ W_n & M_n \end{pmatrix}.$$

The rest of the proof of this Step can be established by replicating the appropriate decompositions $n-1$ times, as demonstrated in the Proof of Theorem 1. T-DEC is applicable at all of the following steps as all constituent subsocieties will be of W-type by the fact that $r_i > r_{i+1}$ for all $i < n$, by stochastic dominance, Lemma 4, NORM and Lemma 3 as demonstrated above. Then, as established in the proof of Theorem 2, we end up with

$$H(S) = \left[\sum_1^n \left(\frac{W_i}{W}\frac{\sum_{i+1}^n M_j}{M} - \frac{M_i}{M}\frac{\sum_{i+1}^n W_j}{W}\right)\right] H\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

concluding Step 1 -notice that the term inside the brackets is positive. In Step 2, we derive the functional form of H for M-perfect societies: S is said to be M-perfect if $M_i > W_i > 0$

for all i and $0 < r_1 < r_2 < \ldots < r_I < \infty$.

Step 2: Take $\mathbf{S} \in C$ that is M-perfect: Consider $S' = (\mathbf{W}', \mathbf{M}')$ with $\mathbf{W}' = \mathbf{M}$ and $\mathbf{M}' = \mathbf{W}$. Thus S' is W-perfect. By SYM, $H(S') = H(S)$. But then, by Step 1, we have

$$
\begin{aligned}
H(S) &= H(S') \\
&= \left[\sum_{1}^{n} \left(\frac{W_i'}{W'} \frac{\sum_{i+1}^{n} M_j'}{M'} - \frac{M_i'}{M'} \frac{\sum_{i+1}^{n} W_j'}{W'} \right) \right] H \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \left[\sum_{1}^{n} \left(\frac{M_i}{M} \frac{\sum_{i+1}^{n} W_j}{W} - \frac{W_i}{W} \frac{\sum_{i+1}^{n} M_j}{M} \right) \right] H \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= - \left[\sum_{1}^{n} \left(\frac{W_i}{W} \frac{\sum_{i+1}^{n} M_j}{M} - \frac{M_i}{M} \frac{\sum_{i+1}^{n} W_j}{W} \right) \right] H \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

concluding Step 2 -notice that the term inside the brackets in the last line is negative. Hence we have driven the functional form of H for very specific types of societies. In the following two steps we show that this functional form actually extends to any society of any type.

Step 3: Take $\mathbf{S} \in C$ of W-type:

First let $S = (\mathbf{W}, \mathbf{M})$ be such that $M_i \geq 1$ for all i and let $T = W + M$. Consider $S' = (\mathbf{W}', \mathbf{M}') = (\mathbf{W}', \mathbf{M})$ and $S'' = (\mathbf{W}'', \mathbf{M}'') = (\mathbf{W} + \mathbf{W}', \mathbf{M})$, where $\mathbf{W}'$ is such that for all i, $W_i' = \sum_{k=2}^{n+2-i} T^k$.

We will first show that both S' and S'' are W-perfect and hence, of W-type. Let us start with S': Notice that for all i, we have $W_i' > M_i' > 0$. S' is W-perfect since for any $i < n$, $r_i' > r_{i+1}'$ holds as shown below

$$
\begin{aligned}
r_i' = \frac{W_i'}{M_i'} &> \frac{W_{i+1}'}{M_{i+1}'} = r_{i+1}' \\
\frac{\sum_{k=2}^{n+2-i} T^k}{M_i} &> \frac{\sum_{k=2}^{n+1-i} T^k}{M_{i+1}} \\
\frac{T^2 + T \sum_{k=2}^{n+1-i} T^k}{M_i} > \frac{T \sum_{k=2}^{n+1-i} T^k}{M_i} &> \frac{\sum_{k=2}^{n+1-i} T^k}{M_{i+1}} \\
\frac{T}{M_i} &> \frac{1}{M_{i+1}}
\end{aligned}
$$

follows from $M_i \geq 1$ for all i, establishing W-perfectness of S'. Now consider S'' as defined.

For any i, we have $W_i'' > M_i'' > 0$. For any $i < n$, $r_i''>r_{i+1}''$ holds as shown below

$$\begin{aligned}
r_i'' = \frac{W_i''}{M_i''} &> \frac{W_{i+1}''}{M_{i+1}''} = r_{i+1}'' \\
\frac{W_i + W_i'}{M_i} &> \frac{W_{i+1} + W_{i+1}'}{M_{i+1}} \\
\frac{W_i + \sum_{k=2}^{n+2-i} T^k}{M_i} &> \frac{W_{i+1} + \sum_{k=2}^{n+1-i} T^k}{M_{i+1}} \\
M_{i+1}W_i + M_{i+1} \sum_{k=2}^{n+2-i} T^k &> M_i W_{i+1} + M_i \sum_{k=2}^{n+1-i} T^k \\
M_{i+1}W_i + M_{i+1}T^2 + M_{i+1}T \sum_{k=2}^{n+1-i} T^k &> M_i W_{i+1} + M_i \sum_{k=2}^{n+1-i} T^k
\end{aligned}$$

follows from $M_i \geq 1$ for all i, $M_{i+1}T^2 > M_i W_{i+1}$ and $M_{i+1}T \sum_{k=2}^{n+1-i} T^k > M_i \sum_{k=2}^{n+1-i} T^k$ establishing W-perfectness of S''. As shown in Step 1, any W-perfect S is of W-type. Hence both S' and S'' are of W-type. But then, Step 1 and T-DEC implies

$$\begin{aligned}
H(S'') =& \frac{W}{W+W'} H(S) + \frac{W'}{W+W'} H(S'). \\
H(S) =& \frac{W+W'}{W} H(S'') - \frac{W'}{W} H(S') \\
H(S) =& \frac{W+W'}{W} \left[\sum_1^n \left(\frac{(W_i + W_i')}{W+W'} \frac{\sum_{i+1}^n M_j}{M} - \frac{M_i}{M} \frac{\sum_{i+1}^n (W_j + W_j')}{W+W'} \right) \right] H \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
& = - \frac{W'}{W} \left[\sum_1^n \left(\frac{W_i'}{W'} \frac{\sum_{i+1}^n M_j}{M} - \frac{M_i}{M} \frac{\sum_{i+1}^n W_j'}{W'} \right) \right] H \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
& = \left[\sum_1^n \left(\frac{W_i}{W} \frac{\sum_{i+1}^n M_j}{M} - \frac{M_i}{M} \frac{\sum_{i+1}^n W_j}{W} \right) \right] H \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (27)
\end{aligned}$$

Hence we have established the functional form for any W-type $S \in C$ with $M_i \geq 1$ for all i. Now, take any W-type $S \in C$ with $M_i > 0$ for all i and consider S' such that $\mathbf{W}' = \mathbf{W}$ and $\mathbf{M}' = \frac{1}{\prod_{\{j: M_j < 1\}} M_j} \mathbf{M}$. Hence, $M_i' \geq 1$ for all i. By Lemma 3, $H(S') = H(S)$ and hence

$$
\begin{aligned}
H(S') =& H(S) = \left[\sum_1^n \left(\frac{W_i}{W} \frac{\sum_{i+1}^n M_j}{M} - \frac{M_i}{M} \frac{\sum_{i+1}^n W_j}{W} \right) \right] H \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \left[\sum_1^n \left(\frac{W_i}{W} \frac{\sum_{i+1}^n \frac{1}{\prod_{\{j:M_j<1\}} M_j} M_j}{\frac{1}{\prod_{\{j:M_j<1\}} M_j} M} - \frac{\frac{1}{\prod_{\{j:M_j<1\}} M_j} M_i}{\frac{1}{\prod_{\{j:M_j<1\}} M_j} M} \frac{\sum_{i+1}^n W_j}{W} \right) \right] H \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \left[\sum_1^n \left(\frac{W_i}{W} \frac{\sum_{i+1}^n M'_j}{M'} - \frac{M'_i}{M'} \frac{\sum_{i+1}^n W_j}{W} \right) \right] H \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}
$$

Finally take any W-type $S \in C$. Consider S' such that $\mathbf{W}' = \mathbf{W}$ and for all i with $M_i \neq 0$, $M'_i = M_i$ and for all i with $M_i = 0$, $M'_i = \varepsilon$ for some ε in a small neighborhood of 0. By continuity of H, we have

$$
\begin{aligned}
H(S) &= \lim_{\varepsilon \to 0} H(S') \\
&= \lim_{\varepsilon \to 0} \left[\sum_1^n \left(\frac{W_i}{W} \frac{\sum_{i+1}^n M'_j}{M'} - \frac{M'_i}{M'} \frac{\sum_{i+1}^n W_j}{W} \right) \right] H \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \left[\sum_1^n \left(\frac{W_i}{W} \frac{\sum_{i+1}^n M_j}{M} - \frac{M_i}{M} \frac{\sum_{i+1}^n W_j}{W} \right) \right] H \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

establishing the functional form for any W-type $S \in C$, and concluding Step 3. If instead we consider any M-type $S \in C$, then Step 2 and symmetric argumentation to Step 3 would yield

$$
H(S) = \left[\sum_1^n \left(\frac{M_i}{M} \frac{\sum_{i+1}^n W_j}{W} - \frac{W_i}{W} \frac{\sum_{i+1}^n M_j}{M} \right) \right] H \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
$$

Since by construction each S is of W-type or of M-type, for any society S, we arrive

$$
H(S) = \left| \sum_1^n \left(\frac{W_i}{W} \frac{\sum_{i+1}^n M_j}{M} - \frac{M_i}{M} \frac{\sum_{i+1}^n W_j}{W} \right) \right| H \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
$$

By ST, H is a nonzero function. Thus, $H(1, 0; 0, 1)$ is a strictly positive real number. $\square$